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Isomorphisms in pro-categories

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Abstract

A morphism of a category which is simultaneously an epimorphism and a monomorphism is called a bimorphism. In (Dydak and Ruiz del Portal (Monomorphisms and epimorphisms in pro-categories, preprint)) we gave characterizations of monomorphisms (resp. epimorphisms) in arbitrary pro-categories, pro-C, where C has direct sums (resp. weak push-outs). In this paper, we introduce the notions of strong monomorphism and strong epimorphism. Part of their significance is that they are preserved by functors. These notions and their characterizations lead us to important classical properties and problems in shape and pro-homotopy. For instance, strong epimorphisms allow us to give a categorical point of view of uniform movability and to introduce a new kind of movability, the sequential movability. Strong monomorphisms are connected to a problem of K. Borsuk regarding a descending chain of retracts of ANRs. If $f:X \to Y$ is a bimorphism in the pointed shape category of topological spaces, we prove that f is a weak isomorphism and f is an isomorphism provided Y is sequentially movable and X or Y is the suspension of a topological space. If $f:X \to Y$ is a bimorphism in the pro-category pro-H_0 (consisting of inverse systems in H_0, the homotopy category of pointed connected CW complexes) we show that f is an isomorphism provided Y is sequentially movable.

MSC: 16B50; 18D35; 54C56

1. Introduction

The fundamental problem in any category C is to detect its isomorphisms. A way to do it is by considering, as in the category of groups, the notions of epimorphism and monomorphism in abstract categories.

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Definition 1.1. A morphism $f:X \to Y$ of a category C is called an *epimorphism* if the induced function $f^*:\ \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$ is one-to-one for each object Z of C.

A morphism $f:X \to Y$ of a category C is called a *monomorphism* if the induced function $f_*:\ \mathrm{Mor}(Z,X) \to \mathrm{Mor}(Z,Y)$ is one-to-one for each object Z of C.

In common terms, f is an epimorphism (respectively, monomorphism) of C if $g\circ f = h\circ f$ (respectively, $f\circ g = f\circ h$) implies $g=h$ for any two morphisms $g,h:Y \to Z$ (respectively, $g,h:Z \to X$).

The main drawback of the two concepts is that they are not functorial, i.e. they are not preserved by covariant functors. One can easily check that morphisms f of a category C with the property that $F(f)$ is a monomorphism (respectively, an epimorphism) of the category D for any covariant functor $F:C \to D$ are exactly those having a left (respectively, right) inverse. Having a left (respectively, right) inverse is, obviously, a functorial property.

A well-known and easy exercise is the following.

Proposition 1.2. *A monomorphism* (*respectively*, *epimorphism*) *which has a left* (*respectively*, *right*) *inverse is an isomorphism.*

The main object of our study are isomorphisms in pro-categories (see a review of pro-categories in the next section). In the case of pro-categories one can consider the following variant of functoriality: Suppose f is a morphism of pro-C. When is (pro-F)(f) a monomorphism (respectively, an epimorphism) of pro-D for any covariant functor $F:C \to D$?

It turns out (see 11.6) that those are exactly strong monomorphisms (respectively, strong epimorphisms)—the key concepts for this paper (see Definition 3.2). Our best results characterizing isomorphisms of pro-categories are: Corollary 3.14 (stating that f is an isomorphism of pro-C if and only if it is a strong monomorphism and an epimorphism) and Corollary 3.16 (stating that, for categories C with direct sums, f is an isomorphism of pro-C if and only if it is a strong epimorphism and a monomorphism). Our best general application of strong epimorphisms is a characterization of uniform movability in Proposition 4.2 with the resulting characterization of isomorphisms $f:X \to Y$ such that Y is uniformly movable and $\varprojlim(f)$ is an isomorphism of C (see Corollary 4.4).

Definition 1.3. A morphism $f:X \to Y$ of a category C is called an *bimorphism* if it is both an epimorphism and a monomorphism of C.

A category C is called *balanced* if every bimorphism of C is an isomorphism.

The following question was posed in [8].

Problem 1.4. Suppose a category C is balanced. Is the pro-category pro-C balanced?

This question was answered negatively in [9], so an amended version of it is as follows.

Problem 1.5. Suppose C is balanced category with direct sums and weak push-outs. Is the pro-category pro-C balanced?

A natural question is to decide which common categories are balanced. It is so in the case of the homotopy category H_0 of pointed connected CW complexes. The question of whether H_0 is balanced has been open for a while with Dyer and Roitberg [10] resolving it in positive and Dydak [6] giving a simple proof of it. Mukherjee [18] generalized the approach from [10] to the equivariant case and Morón-Ruiz del Portal [16] showed that the shape category of pointed, movable, metric continua is not balanced but every weak isomorphism is a bimorphism. We recommend [12] for a near complete list and a thorough review of results related to monomorphisms and epimorphisms of H_0.

In [8] the authors embarked on a program to determine if pro-H_0 is balanced and that paper contains results on bimorphisms in tow(H_0), the category of towers in H_0. Section 8 of this paper generalizes those results to bimorphisms of pro-H_0. In Section 9 we investigate bimorphisms of the shape category and in Section 10 we relate the concept of strong monomorphism to a question of K. Borsuk.

2. Review of pro-categories

Let us recall basic facts about pro-categories (for details see [15]). Loosely speaking, the pro-category pro-C of C is the universal category with inverse limits containing C as a full subcategory. Quite often one considers pro-objects indexed by small cofiltered categories. However, those are isomorphic to pro-objects indexed by directed sets (see [15, pp. 14–15]), so in this paper the objects of pro-C are inverse systems $X=(X_\alpha, p_\alpha^\beta, A)$ in C such that A is a directed set. To simplify notation we will call A the *index set* of X and we will denote it by $I(X)$. Given $\alpha,\beta\in I(X)$ with $\alpha<\beta$, the *bonding morphism* p_α^β from X_β to X_α will be denoted by $p(X)_\alpha^\beta$.

Given an inverse system X in C and $P\in \mathrm{Ob}(C)$ (P is an object of C), the set of morphisms of pro-C from X to P is the direct limit of $\mathrm{Mor}(X_\alpha,P)$, $\alpha\in I(X)$. Thus each morphism f from X to P has its *representative* $g:X_\alpha\to P$, and two representatives $g:X_\alpha\to P$ and $h:X_\beta\to P$ determine the same morphism if there is $\gamma>\alpha,\beta$ with $g\circ p(X)_\alpha^\gamma = h\circ p(X)_\beta^\gamma$. In particular, the morphism from X to X_α represented by the identity $X_\alpha\to X_\alpha$ is called the *projection morphism* and denoted by $p(X)_\alpha$. It is clear how to compose morphisms from X to P with morphisms from P to Q if $P,Q\in\mathrm{Ob}(C)$.

If X and Y are two inverse systems in C, then any morphism $f:X\to Y$ of pro-C can be identified with a family of morphisms $\{f_\alpha:X\to Y_\alpha\}_{\alpha\in I(Y)}$ such that $p(Y)_\alpha^\beta\circ f_\beta=f_\alpha$ for all $\alpha<\beta$ in $I(Y)$. Notice that $f_\alpha=p(Y)_\alpha\circ f$ for each $\alpha\in I(Y)$. Therefore one has a simple characterization of isomorphisms of pro-C.

Proposition 2.1. *A morphism $f:X\to Y$ of* pro-C *is an isomorphism if and only if $f^*:\ \mathrm{Mor}(Y,P)\to\mathrm{Mor}(X,P)$ is a bijection for all $P\in\mathrm{Ob}(C)$.*

Of particular interest are isomorphisms $f:X\to P\in\mathrm{Ob}(C)$. If such an isomorphism exists, then X is called *stable*.

If s is a directed subset of $I(X)$, then by X_s we will denote the induced inverse system $(X_\alpha, p(X)^\beta_\alpha, s)$. Notice that the family $\{p_\alpha : X \to X_\alpha\}_{\alpha\in s}$ induces a morphism from X to X_s which will be denoted by $p(X)_s$. If s is a *cofinal* subset of $I(X)$ (that means for any $\alpha \in I(X)$ there is $\beta \in s$ so that $\beta > \alpha$), then $p(X)_s$ is an isomorphism of pro-C.

Of particular use are *level morphisms* of pro-C. Those are $f : X \to Y$, where X and Y have identical index sets and there are representatives $f_\alpha : X_\alpha \to Y_\alpha$ of $p(Y)_\alpha \circ f$, $\alpha \in I(X)$, such that $p(Y)^\beta_\alpha \circ f_\beta = f_\alpha \circ p(X)^\beta_\alpha$ for all $\alpha < \beta$. In such a case we say that f is a *level morphism induced by the family* $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha\in I(Y)}$.

It is also convenient to consider inverse systems X such that $I(X)$ is a *cofinite directed set* which means that each element of $I(X)$ has only finitely many predecessors. The following result is of particular use (see [15, Theorem 3, p. 12]).

Proposition 2.2. *For any morphism $f : X \to Y$ of* pro-C *there exists a level morphism $f' : X' \to Y'$ and isomorphisms $i : X \to X'$, $j : Y' \to Y$ such that $f = j \circ f' \circ i$ and $I(X')$ is a cofinite directed set. Moreover, the bonding morphisms of X' (respectively, Y') are chosen from the set of bonding morphisms of X (respectively, Y).*

In the special case of X being an object of C one can create X' by putting $X'_\alpha = X$ and $p(X')^\beta_\alpha = id_X$ for each $\beta > \alpha$ in $I(Y)$. Notice that in this case $Y' = Y$ and f' is induced by the family $\{p(Y)_\alpha \circ f\}_{\alpha\in I(Y)}$. In what follows morphisms from objects X of C to inverse systems Y in C will be automatically replaced by level morphisms from X' to Y. This is needed as part of our strategy is to select increasing sequences in $I(X)$ which is not possible if $I(X)$ contains an upper bound (which implies that X is *stable*, i.e. isomorphic to an object of C).

Another reason level morphisms are very useful is that one has a very simple criterion of them being an isomorphism (see [15, Theorem 5, p. 112]).

Proposition 2.3. *A level morphism $f : X \to Y$ of* pro-C *is an isomorphism if and only if for each $\alpha \in I(X)$ there is $\beta > \alpha$ and $g : Y_\beta \to X_\alpha$ such that $f_\alpha \circ g = p(Y)^\beta_\alpha$ and $g \circ f_\beta = p(X)^\beta_\alpha$.*

We will need the following characterization of monomorphisms in pro-C such that C has direct sums (see [9, Proposition 2.3]).

Proposition 2.4. *Suppose that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha\in I(X)}$ is a level morphism of* pro-C. *Consider the following conditions*:

(a) *f is a monomorphism.*
(b) *For each $\alpha \in I(X)$ there is $\beta \in I(X)$, $\beta > \alpha$, such that for any $u, v : P \in \mathrm{Ob}(C) \to X_\beta$, $f_\beta \circ u = f_\beta \circ v$ implies that $p(X)^\beta_\alpha \circ u = p(X)^\beta_\alpha \circ v$.*
(c) *For each $\alpha \in I(X)$ there is $\beta \in I(X)$, $\beta > \alpha$, such that for any $u, v : T \to X_\beta$, $f_\beta \circ u = f_\beta \circ v$ implies that $p(X)^\beta_\alpha \circ u = p(X)^\beta_\alpha \circ v$.*

Conditions (b) *and* (c) *are equivalent and imply Condition* (a). *If C has direct sums, then all three conditions are equivalent*

Also, we will give a characterization of epimorphisms in pro-C such that C has weak push-outs.

Definition 2.5. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & P \end{tikzcd}$$

in category C is a *weak push-out* (respectively, *push-out*) of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

if for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u'"] \\ C \arrow[r, "v'"] & P' \end{tikzcd}$$

in C there is a morphism (respectively, a unique morphism) $t : P \to P'$ such that $u' = t \circ u$ and $v' = t \circ v$. If every diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

in C has a weak push-out, then we say that C is a category with *weak push-outs*.

Proposition 2.6. *The homotopy category H_0 of pointed connected CW complexes is a category with weak push-outs but not a category with push-outs.*

Proof. Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

is a commutative diagram in H_0. Pick cellular maps $b: A \to B$ and $c: A \to C$ representing f and g, respectively. Let P be the union of the reduced mapping cylinders $M(a)$ and $M(b)$ so that $M(a) \cap M(b) = A$. There are natural inclusions $u: C \to P$ and $v: B \to P$. Given maps $u': C \to P'$ and $v': B \to P'$ so that $u' \circ c$ is homotopic to $v' \circ b$, any homotopy H from $u' \circ c$ to $v' \circ b$ leads naturally to a map $t: P \to P'$ so that t extends both u' and v'.

To show that H_0 does not have push-outs, let us use an example provided to us by the referee. Namely, both A and C are the unit circle S^1, B is trivial, and $g(z) = z^2$ for $z \in S^1$. Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & P \end{tikzcd}$$

is a push-out diagram. Given any pointed CW complex X and an element $\alpha \in \pi_1(X)$ satisfying $\alpha^2 = 1$, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "const"] \\ C \arrow[r, "\alpha"] & X \end{tikzcd}$$

is commutative, so there is $h_\alpha: P \to X$ with $\alpha = h_\alpha \circ u$. That correspondence between α and h_α establishes a bijection between morphisms from P to X and the subset of $\pi_1(X)$ consisting of elements whose square is 1. In particular, all higher cohomology of P vanishes and $H^1(P;G)$ consists of elements of G order two or less. The contradiction arises by looking at the exact sequence $0 \to Z/2 \to Z/4 \to Z/2 \to 0$ and induced exact sequence of cohomology groups (recall that higher cohomology of P vanishes) $0 \to H^1(P;Z/2) \to H^1(P;Z/4) \to H^1(P;Z/2) \to 0$. It contradicts that $H^1(P;G) = \{g \in G | 2g = 0\}$ as there is no exact sequence $0 \to Z/2 \to Z/2 \to Z/2 \to 0$. □

Proposition 2.7. *Suppose $f = \{f_\alpha: X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of* pro-C *and consider the following conditions*:

(a) *for each $\alpha \in I(X)$ there is $\beta \in I(X)$, $\beta > \alpha$, such that for any $u, v: Y_\alpha \to P \in \mathrm{Ob}(C)$, $u \circ f_\alpha = v \circ f_\alpha$ implies that $u \circ p(Y)_\alpha^\beta = v \circ p(Y)_\alpha^\beta$.*
(b) *f is an epimorphism of* pro-C.

Condition (a) *is stronger than Condition* (b). *If C is a category with weak push-outs, then both conditions are equivalent.*

Proof. (a) ⇒ (b). It suffices to show that $u, v: Y \to P \in \mathrm{Ob}(C)$ and $u \circ f = v \circ f$ implies $u = v$. Pick representatives $u': Y_\alpha \to P$ of u and $v': Y_\alpha \to P$ of v for some $\alpha \in I(X)$. We may assume that $u' \circ f_\alpha = v' \circ f_\alpha$. There is $\beta > \alpha$ such that $u' \circ p(Y)_\alpha^\beta = v' \circ p(Y)_\alpha^\beta$ which implies $u = v$.

(b) $\Rightarrow$ (a). If C is a category with weak push-outs. Let

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "f_\alpha"'] & Y_\alpha \arrow[d, "a"] \\ Y_\alpha \arrow[r, "b"] & M \end{tikzcd}$$

be a weak push-out of

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "f_\alpha"'] & Y_\alpha \\ Y_\alpha & \end{tikzcd}$$

There is $\beta > \alpha$ so that $a \circ p(Y)_\alpha^\beta = b \circ p(Y)_\alpha^\beta$ as f is an epimorphism. If $u, v : Y_\alpha \to P \in \mathrm{Ob}(C)$, $u \circ f_\alpha = v \circ f_\alpha$, then

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "f_\alpha"'] & Y_\alpha \arrow[d, "u"] \\ Y_\alpha \arrow[r, "v"] & P \end{tikzcd}$$

is commutative, so there is $i : M \to P$ such that $i \circ b = v$ and $i \circ a = u$.

Therefore,

$$u \circ p(Y)_\alpha^\beta = i \circ a \circ p(Y)_\alpha^\beta = i \circ b \circ p(Y)_\alpha^\beta = v \circ p(Y)_\alpha^\beta. \qquad \square$$

Remark 2.8. In the original version of this paper Proposition 2.7 was stated and used for categories C with push-outs. However, our main application was for $C = H_0$ and it was pointed out by the referee that H_0 does not have push-outs (see the proof of Proposition 2.6). That is how the notion of weak push-outs was created to salvage 2.7 and allow all applications to be valid.

Let us proceed with an abstract construction. Its meaning will be explained shortly.

Definition 2.9. Suppose c is an ordinal. Given $X \in \mathrm{Ob}(\text{pro-}C)$ an object $\mathrm{Sub}_c(X)$ of pro-(pro-C) is defined as follows:

(1) the index set $I(\mathrm{Sub}_c(X))$ of $\mathrm{Sub}_c(X)$ consists of all increasing functions $s : \{n \mid n < c\} \to I(X)$, where n is a cardinal number smaller than c,
(2) $s \leqslant t$, $s, t \in I(\mathrm{Sub}_c(X))$, holds if and only if $s(n) \leqslant t(n)$ for all $n < c$,
(3) $\mathrm{Sub}_c(X)_s := (X_{s(n)}, p(X)_{s(n)}^{s(m)})$ for all $s \in I(\mathrm{Sub}_c(X))$,
(4) $p(\mathrm{Sub}_c(X))_s^t$ is the level morphism induced by $\{p(X)_{s(n)}^{t(n)}\}_{n<c}$ for all $s < t$.

An important case of Definition 2.9 is $c=2$: $\mathrm{Sub}_2(X)$ is another way of looking at X as an object of pro-(pro-C) (the standard way corresponds to the canonical embedding of any category into its pro-category). Observe that the projections $\mathrm{Sub}_c(X)_s \to X_{s(1)}$ induce a morphism from $\mathrm{Sub}_c(X)$ to $\mathrm{Sub}_2(X)$ for any $c \geqslant 2$. Also notice that the family $\{p(X)_\alpha : X \to X_\alpha\}_{\alpha \in I(X)}$ induces a morphism from X to $\mathrm{Sub}_2(X)$ of pro-(pro-C). That morphism will be used later on to explain the concept of uniform movability.

Another important case of Definition 2.9 is $c=\omega_0$ as part of our work is related to reducing properties of pro-C to the properties of its full subcategory tow(C) consisting of *towers*, i.e. inverse sequences in C.

Just as every morphism of pro-C from X to an object of C factors through a subterm of X, every morphism from X to a tower factors through a subtower.

Proposition 2.10. *Suppose C is a category. If $f:X \to Y$ is a morphism to a tower, then there is a subtower X_s of X and a level morphism $g:X_s \to Y$ such that $f = g \circ p(X)_s$.*

Proof. Choose $s(1) \in I(X)$ such that there is a representative $g_1 : X_{s(1)} \to Y_1$ of $p(Y)_1 \circ f$. Suppose $s(i)$ and $g_i : X_{s(i)} \to Y_i$ are defined for $i \leqslant n$ such that g_i is a representative of $p(Y)_i \circ f$. Find $\alpha \in I(X)$, $\alpha > s(n)$ such that there is a representative $h : X_\alpha \to Y_{n+1}$ of $p(Y)_{n+1} \circ f$. Since both g_n and $p(Y)^{n+1}_n \circ h$ are representatives of $p(Y)_n \circ f$, there is $s(n+1) > \alpha$ such that $g_n \circ p(X)^{s(n+1)}_{s(n)} = p(Y)^{n+1}_n \circ h \circ p(X)^{s(n+1)}_\alpha$. By putting $g_{n+1} = h \circ p(X)^{s(n+1)}_\alpha$ we complete the inductive construction of a level morphism $g : X_s \to Y$ satisfying $g \circ p(X)_s = f$. $\square$

Proposition 2.11. *Suppose C is a category and $f: Y \to Z$ is a morphism of towers in C. If X_s is a subtower of X and $g,h : X_s \to Y$ are two morphisms such that $f \circ g \circ p(X)_s = f \circ h \circ p(X)_s$, then there is a subtower X_t of X such that $t > s$ and $f \circ g \circ p(X)^t_s = f \circ h \circ p(X)^t_s$.*

Proof. *Special case:* f, g, and h are level morphisms.

For each $n \in N$ the morphisms $f_n \circ g_n$ and $f_n \circ h_n$ are representatives of $p(Z)_n \circ f \circ g \circ p(X)_s$, so there is $t(n) > s(n)$ such that $f_n \circ g_n \circ p(X)^{t(n)}_{s(n)} = f_n \circ h_n \circ p(X)^{t(n)}_{s(n)}$. Using induction one can ensure $t(n) > t(n-1)$ which completes the proof of Special Case.

General case: By Proposition 2.10 there is an increasing sequence $u : N \to N$ and a level morphism $f' : Y_u \to Z$ so that $f = f' \circ p(Y)_u$. Using Proposition 2.10 again one can find an increasing sequence $v : N \to \mathrm{im}(s)$ and level morphisms $g', h' : X_v \to Y_u$ such that $g' \circ p(X_s)_v = p(Y)_u \circ g$ and $h' \circ p(X_s)_v = p(Y)_u \circ h$. Since $f' \circ g' \circ p(X)_v = f' \circ g' \circ p(X_s)_v \circ p(X)_s = f' \circ p(Y)_u \circ g \circ p(X)_s = f \circ g \circ p(X)_s$ and, similarly, $f' \circ h' \circ p(X)_v = f \circ h \circ p(X)_s$, we get $f' \circ g' \circ p(X)_v = f' \circ h' \circ p(X)_v$. By Special Case there is $t > v$ so that $f' \circ g' \circ p(X)^t_v = f' \circ h' \circ p(X)^t_v$. Now $f \circ g \circ p(X)^t_s = f' \circ p(Y)_u \circ g \circ p(X)^t_s = f' \circ g' \circ p(X_s)_v \circ p(X)^t_s = f' \circ g' \circ p(X)^t_v$ and, similarly, $f \circ h \circ p(X)^t_s = f' \circ h' \circ p(X)^t_v$. Therefore, $f \circ g \circ p(X)^t_s = f \circ h \circ p(X)^t_s$. $\square$

Corollary 2.12. *Let C be a category.*

(1) *Every monomorphism (respectively, epimorphism) of C is a monomorphism (respectively, epimorphism) of* pro-C.
(2) *Every monomorphism (respectively, epimorphism) of* tow(C) *is a monomorphism (respectively, epimorphism) of* pro-C.
(3) *Every bimorphism of C or* tow(C) *is a bimorphism of* pro-C.

Proof. (A) Let us prove (1) and (2) for epimorphisms. Suppose $f:X \to Y$ is an epimorphism of D, $D=C$ or $D=\mathrm{tow}(C)$, and $g,h:Y\to Z$ satisfy $g\circ f=h\circ f$. To show $g=h$ it suffices to prove $p(Z)_\alpha\circ g=p(Z)_\alpha\circ h$ for all $\alpha\in I(Z)$. Since Z_α is an object of D and $(p(Z)_\alpha\circ g)\circ f=(p(Z)_\alpha\circ h)\circ f$, one gets $p(Z)_\alpha\circ g=p(Z)_\alpha\circ h$ as f is an epimorphism of D.

(B) A. Let us prove (1) and (2) for monomorphisms. Suppose $f:X\to Y$ is a monomorphism of D, $D=C$ or $D=\mathrm{tow}(C)$, and $g,h:Z\to X'$ satisfy $f\circ g=f\circ h$. By Propositions 2.10 and 2.11 there is a sequence s in $I(X)$ (an element s of $I(X)$ if $D=C$) and there are level morphisms $g_s,h_s:Z_s\to X$ such that $g=g_s\circ p(Z)_s$, $h=h_s\circ p(Z)_s$ and $f\circ g_s=f\circ h_s$. Since f is a monomorphism of D, $g_s=h_s$ which implies $g=h$.

(C) The proof of (3) follows directly from (1) and (2). □

Proposition 2.13. *Suppose $f:X\to Y$ is a level morphism of* pro-C *and Z is an inverse system in C. Let Σ be the set of sequences s in $I(X)$ such that $(f_s)_*:\mathrm{Mor}(Z,X_s)\to\mathrm{Mor}(Z,Y_s)$ is a bijection. If Σ is cofinal in the set of all sequences in $I(X)$, then $f_*:\mathrm{Mor}(Z,X)\to\mathrm{Mor}(Z,Y)$ is a bijection.*

Proof. Notice that it suffices to show that f_* is a surjection. Given $g:Z\to Y$ and $\alpha\in I(X)$ define $h_\alpha:Z\to X_\alpha$ as follows:

(1) Find a sequence s in $I(X)$ such that $s(1)>\alpha$ and $(f_s)_*:\mathrm{Mor}(Z,X_s)\to\mathrm{Mor}(Z,Y_s)$ is a bijection.
(2) Pick $k_s:Z\to X_s$ with $f_s\circ k_s=p(Y)_s\circ g$.
(3) Define h_α as $p(X)_\alpha^{s(1)}\circ p(X_s)_1\circ k_s$.

Our first observation is that the above definition does not depend on s. Indeed, if $t>s$, then $f_s\circ(p(X)_s^t\circ k_t)=p(Y)_s^t\circ f_t\circ k_t=p(Y)_s^t\circ p(Y)_t\circ g=p(Y)_s\circ g=f_s\circ k_s$. Therefore $p(X)_s^t\circ k_t=k_s$ and $p(X)_\alpha^{s(1)}\circ p(X_s)_1\circ k_s=p(X)_\alpha^{s(1)}\circ p(X_s)_1\circ p(X)_s^t\circ k_t=p(X)_\alpha^{t(1)}\circ p(X_t)_1\circ k_t$.

Using the first observation and given $\alpha<\beta$ one can find the same sequence s to define both h_α and h_β. Now, $p(X)_\alpha^\beta\circ h_\beta=p(X)_\alpha^\beta\circ p(X)_\beta^{s(1)}\circ p(X_s)_1\circ k_s=p(X)_\alpha^{s(1)}\circ p(X_s)_1\circ k_s=h_\alpha$. That means $\{h_\alpha\}_{\alpha\in I(X)}$ is a morphism h from Z to X.

Finally, for all $\alpha\in I(X)$, $p(Y)_\alpha\circ(f\circ h)=f_\alpha\circ h_\alpha=f_\alpha\circ p(X)_\alpha^{s(1)}\circ p(X_s)_1\circ k_s=p(Y)_\alpha^{s(1)}\circ f_{s(1)}\circ p(X_s)_1\circ k_s=p(Y)_\alpha^{s(1)}\circ p(Y_s)_1\circ(f_s\circ k_s)=p(Y)_\alpha^{s(1)}\circ p(Y_s)_1\circ p(Y)_s\circ g=p(Y)_\alpha\circ g$. That means $f\circ h=g$. □

3. Strong monomorphisms and strong epimorphisms

Unless stated otherwise, C is an arbitrary category in this section.

The following characterization of isomorphisms in pro-categories is useful in introducing and understanding of the main concepts of this section; strong monomorphisms and strong epimorphisms.

Proposition 3.1. *Let $f:X\to Y$ be a morphism in* pro-C. *f is an isomorphism if and only if for any $P,Q\in\mathrm{Ob}(C)$ and any commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

there exists $u:Y\to P$ such that $g\circ u=b$ and $u\circ f=a$.

Proof. If f^{-1} exists, then clearly $u=a\circ f^{-1}$ satisfies the desired equalities.

Suppose morphism u exists for any commutative diagram. Without loss of generality, we may assume that $f=\{f_\alpha:X_\alpha\to Y_\alpha\}_{\alpha\in I(X)}$ is a level morphism of pro-C from X to Y. Since

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p(X)_\alpha}\downarrow & & \downarrow{\scriptstyle p(Y)_\alpha} \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

is commutative for any $\alpha\in A$, there is $u_\alpha:Y\to X_\alpha$ such that $u_\alpha\circ f=p_\alpha$ and $f_\alpha\circ u_\alpha=p(Y)_\alpha$. To prove that f is an isomorphism it suffices to show that $f^*:\mathrm{Mor}(Y,P)\to\mathrm{Mor}(X,P)$ is a bijection for each $P\in\mathrm{Ob}(C)$ (see Proposition 2.1). Since any $g:X\to P$ factors as $g=g'\circ p(X)_\alpha$ for some $\alpha\in I(X)$, putting $h=g'\circ u_\alpha$ one gets $h\circ f=g$, i.e. f^* is a surjection. If $g,h:Y\to P\in\mathrm{Ob}(C)$ satisfy $f\circ g=f\circ h$, then one can find representatives $g',h':Y_\alpha\to P$ of g and h such that $g'\circ f_\alpha=h'\circ f_\alpha$. Now $g=g'\circ q_\alpha=g'\circ f_\alpha\circ u_\alpha=h'\circ f_\alpha\circ u_\alpha=h'\circ p(Y)_\alpha=h$, i.e. f^* is an injection. □

Definition 3.2. A morphism $f:X\to Y$ in pro-C is called a *strong monomorphism* (*strong epimorphism*, respectively) if every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

where $P,Q\in\mathrm{Ob}(C)$, admits a morphism $u:Y\to P$ so that $u\circ f=a$ ($g\circ u=b$, respectively).

Remark 3.3. Notice that, provided C has a terminal object $*$, the object Q in the above diagram is irrelevant (in the case of strong monomorphisms) as it can always be replaced by such $*$.

Remark 3.4. If X and Y are objects of C, then $f:X \to Y$ is a strong monomorphism (strong epimorphism, respectively) if and only if f has a left inverse (a right inverse, respectively). Simply put $g = f$, $a = id_X$, and $b = id_Y$ in the above diagram.

Remark 3.5. Later on (see Remarks 4.3 and 4.18) we will see examples of strong monomorphisms (respectively, strong epimorphisms) in the category pro-*Gr* of pro-groups which do not have a left (respectively, right) inverse.

Proposition 3.6. *Suppose that* $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ *is a level morphism of* pro-C *from* X *to* Y. *For any commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

we may find $\alpha \in I(X)$ *and representatives* $a_\alpha : X_\alpha \to P$ *of* a *and* $b_\alpha : Y_\alpha \to Q$ *of* b *such that*

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "a_\alpha"'] & Y_\alpha \arrow[d, "b_\alpha"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

is commutative.

Proof. Choose representatives $u : X_\beta \to P$ of a and $v : Y_\beta \to Q$ of b. Since $g \circ u \circ p(X)_\beta = g \circ a = b \circ f = v \circ p(Y)_\beta \circ f = v \circ f_\beta \circ p(X)_\beta$, there is $\alpha > \beta$ such that $g \circ u \circ p(X)^\alpha_\beta = v \circ f_\beta \circ p(X)^\alpha_\beta$. Put $a_\alpha = u \circ p(X)^\alpha_\beta$ and $b_\alpha = v \circ p(Y)^\alpha_\beta$. □

The following characterization of strong monomorphisms and strong epimorphisms is especially useful. Its immediate consequence is that both properties are preserved by functors pro-F if $F : C \to D$.

Proposition 3.7. *Suppose that* $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ *is a level morphism of* pro-C. *The following statements are equivalent*:

(a) f *is a strong monomorphism* (*strong epimorphism*, *respectively*).
(b) *For each* $\alpha \in I(X)$ *there is a morphism* $u_\alpha : Y \to X_\alpha$ *such that* $u_\alpha \circ f = p(X)_\alpha$ ($f_\alpha \circ u_\alpha = p(Y)_\alpha$, *respectively*).
(c) *For each* $\alpha \in I(X)$ *there is* $\beta \in I(X)$, $\beta > \alpha$ *and a morphism* $g_{\alpha,\beta} : Y_\beta \to X_\alpha$ *such that* $g_{\alpha,\beta} \circ f_\beta = p(X)^\beta_\alpha$ ($f_\alpha \circ g_{\alpha,\beta} = p(Y)^\beta_\alpha$, *respectively*).

Proof. (a) $\Rightarrow$ (b) follows from the definition of strong monomorphisms (strong epimorphisms, respectively).

(b) $\Rightarrow$ (c). u_α has a representative $v : Y_\gamma \to X_\alpha$ for some $\gamma > \alpha$. Since $u_\alpha \circ f = p(X)_\alpha$ ($f_\alpha \circ u_\alpha = p(Y)_\alpha$, respectively), $v \circ f_\gamma$ and $p(X)^\gamma_\alpha$ ($f_\alpha \circ v$ and $p(Y)^\gamma_\alpha$, respectively) are representatives of the same morphism from X to X_α (Y to Y_α, respectively), so there is $\beta > \gamma$ such that $v \circ f_\gamma \circ p(X)^\beta_\gamma = p^\gamma_\alpha \circ p(X)^\beta_\gamma$ ($f_\alpha \circ v \circ p(Y)^\beta_\gamma = p(Y)^\gamma_\alpha \circ p(Y)^\beta_\gamma$, respectively). Put $g_{\alpha,\beta} = v \circ p(Y)^\beta_\gamma$.

(c) $\Rightarrow$ (a). Given a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

we may find $\alpha \in I(X)$ and representatives $a_\alpha : X_\alpha \to P$ and $b_\alpha : Y_\alpha \to Q$ such that

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "a_\alpha"'] & Y_\alpha \arrow[d, "b_\alpha"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

is commutative (see Proposition 3.6). Let $g_{\alpha,\beta} : Y_\beta \to X_\alpha$ be as in (c) and define $u = a_\alpha \circ g_{\alpha,\beta} \circ p(Y)_\beta : Y \to P$. Then, in the case of f being a strong monomorphism,

$$\begin{aligned} u \circ f &= a_\alpha \circ g_{\alpha,\beta} \circ p(Y)_\beta \circ f = a_\alpha \circ g_{\alpha,\beta} \circ f_\beta \circ p(X)_\beta \\ &= a_\alpha \circ p(X)^\beta_\alpha \circ p(X)_\beta = a_\alpha \circ p(X)_\alpha = a. \end{aligned}$$

Similarly, in the case of f being a strong epimorphism,

$$\begin{aligned} g \circ u &= g \circ a_\alpha \circ g_{\alpha,\beta} \circ p(Y)_\beta = b_\alpha \circ f_\alpha \circ g_{\alpha,\beta} \circ p(Y)_\beta \\ &= b_\alpha \circ p(Y)^\beta_\alpha \circ p(Y)_\beta = b_\alpha \circ p(Y)_\alpha = b. \qquad \square \end{aligned}$$

Remark 3.8. In view of Proposition 3.7 one can relate strong epimorphisms and strong monomorphisms to the following concepts previously discussed in literature:

1. *Weak dominations* introduced by Dydak [5] (see also [15, p. 186]) are precisely strong epimorphisms of pro-H_0.

2. Given a compact subset X of the Hilbert cube Q one considers the system $N(X)$ of neighborhoods of X in Q. It is an object of pro-T, where T is the category of topological spaces, and one has a natural morphism $i : X \to N(X)$ of pro-T. Notice that i is a strong monomorphism of pro-T if and only if X is an ANR.

3. The above morphism $i : X \to N(X)$ can be interpreted as a morphism of pro-HT, where HT is the homotopy category of topological spaces. Notice that i is a strong epimorphism of pro-HT if and only if X is *internally movable* (see [1]). Indeed, X is internally movable if for every neighborhood U of X in Q there is a neighborhood V of X in U and a map $r : V \to X$ which is homotopic in U to the inclusion $V \to U$.

4. *Approximate ANRs in the sense of Clapp* [4] are introduced in a way related to strong monomorphisms. Recall that $X \in AANR_C$ if for each $\varepsilon > 0$ there is a neighborhood U of X in Q and a map $r : U \to X$ such that $r|U$ is ε-close to id_X. Also, *approximate ANRs in the sense of Noguchi* [19] and AWNRs of [1,23] are defined in a way resembling strong monomorphisms.

Proposition 3.9. *Let $f : X \to Y$ be a morphism in* pro-C. *The following statements are equivalent*:

(1) *f is a strong monomorphism.*
(2) *$f^* : \mathrm{Mor}(Y,P) \to \mathrm{Mor}(X,P)$ is a surjection for each $P \in \mathrm{Ob}(C)$.*

Proof. Without loss of generality we may assume that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-C from X to Y.

(1) $\Rightarrow$ (2). Given $g : X \to P \in \mathrm{Ob}(C)$ there is a representative $v : X_\alpha \to P$ of g. By Proposition 3.7 there is $u : Y \to X_\alpha$ such that $u \circ f = p(X)_\alpha$. Put $h = v \circ u : Y \to P$. Now $h \circ f = v \circ u \circ f = v \circ p(X)_\alpha = g$.

(2) $\Rightarrow$ (1). Given $\alpha \in A$ there is $u : Y \to X_\alpha$ such that $u \circ f = p(X)_\alpha$. By Proposition 3.7, f is a strong monomorphism. □

Corollary 3.10. *If f is a strong monomorphism* (*strong epimorphism*, *respectively*) *of* pro-C, *then f is a monomorphism* (*epimorphism*, *respectively*) *of* pro-C.

Proof. Without loss of generality we may assume that $f = \{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in I(X)}$ is a level morphism of pro-C from X to Y.

Suppose f is a strong monomorphism and $a, b : Z \to X$ are two morphisms of pro-C such that $f \circ a = f \circ b$. To show $a = b$ it suffices to prove $p(X)_\alpha \circ a = p(X)_\alpha \circ b$ for all $\alpha \in I(X)$. Choose $u : Y \to X_\alpha$ such that $u \circ f = p(X)_\alpha$ (see Proposition 3.7). Now $p(X)_\alpha \circ a = u \circ f \circ a = u \circ f \circ b = p(X)_\alpha \circ b$.

Suppose f is a strong epimorphism and $a, b : Y \to Z$ are two morphisms of pro-C such that $a \circ f = b \circ f$.

Special Case: $Z \in \mathrm{Ob}(C)$. Choose representatives $a', b' : Y_\alpha \to Z$ of a and b, respectively, such that $a' \circ f_\alpha = b' \circ f_\alpha$. By Proposition 3.7 there is $\beta > \alpha$ and $g : Y_\beta \to X_\alpha$ such that $f_\alpha \circ g = p(Y)^\beta_\alpha$. Therefore $a' \circ p(Y)^\beta_\alpha = a' \circ f_\alpha \circ g = b' \circ f_\alpha \circ g = b' \circ p(Y)^\beta_\alpha$ which proves $a = b$.

General Case: To show $a = b$ we need $p(Z)_i \circ a = p(Z)_i \circ b$ for all $i \in I(Z)$ which follows from Special Case. □

Remark 3.11. Notice that there are monomorphisms (respectively, epimorphisms) of pro-*Gr* that are not strong monomorphisms (respectively, strong epimorphisms). Easy examples are the inclusion $Z \to Q$ from integers to rational numbers and the projection $Z \to Z/2$ from integers to integers modulo 2.

Corollary 3.12. *If $g \circ f$ is a strong monomorphism* (*strong epimorphism*, *respectively*), *then f is a strong monomorphism* (*g is a strong epimorphism*, *respectively*).

Proof. Assume $f:X \to Y$ and $g:Y \to Z$.

Suppose $g \circ f$ is a strong monomorphism and $a:X \to P \in \mathrm{Ob}(C)$. By Proposition 3.9 there is $b:Z \to P$ such that $a = b \circ (g \circ f)$. Now, $c = b \circ g$ satisfies $a = c \circ f$ and Proposition 3.9 says that f is a strong monomorphism.

Suppose $g \circ f$ is a strong epimorphism and

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "a"'] & Z \arrow[d, "b"] \\ P \arrow[r, "h"] & Q \end{tikzcd}$$

is a commutative diagram in pro-C with $P, Q \in \mathrm{Ob}(C)$. Since

$$\begin{tikzcd} X \arrow[r, "g \circ f"] \arrow[d, "a \circ f"'] & Z \arrow[d, "b"] \\ P \arrow[r, "h"] & Q \end{tikzcd}$$

is commutative, there is $u:Z \to P$ with $h \circ u = b$ which proves that g is a strong epimorphism. $\square$

The following is the main property of strong monomorphisms.

Theorem 3.13. *Suppose*

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram in pro-C. *If f' is an epimorphism and f is a strong monomorphism, then there is a unique filler $u:T \to X$, i.e. a morphism u such that $g' = u \circ f'$ and $f \circ u = g$.*

Proof. Since f' is an epimorphism, it suffices to prove existence of u. Also, it suffices to prove that $g' = u \circ f'$. Indeed, $g' = u \circ f'$ implies $(f \circ u) \circ f' = f \circ g' = g \circ f'$, so $f \circ u = g$ as f' is an epimorphism.

Given $\alpha \in I(X)$ there is $r(\alpha): Y \to X_\alpha$ such that $r(\alpha) \circ f = p(X)_\alpha$. Put $u_\alpha = r(\alpha) \circ g$. If $\beta > \alpha$, then $p(X)_\alpha^\beta \circ u_\beta \circ f' = p(X)_\alpha^\beta \circ r(\beta) \circ g \circ f' = p(X)_\alpha^\beta \circ r(\beta) \circ f \circ g' = p(X)_\alpha^\beta \circ p(X)_\beta \circ g' = p(X)_\alpha \circ g'$. Similarly, $u_\alpha \circ f' = p(X)_\alpha \circ g'$. Since f' is an epimorphism, $u_\alpha = p(X)_\alpha^\beta \circ u_\beta$ which means that $\{u_\alpha\}_{\alpha \in I(X)}$ is a morphism from T to X. Also, $u_\alpha \circ f' = p(X)_\alpha \circ g'$ for all $\alpha \in I(X)$ means $u \circ f' = g'$. $\square$

Corollary 3.14. *Let $f:X \to Y$ be a morphism in* pro-C. *The following statements are equivalent*:

1. *f is an isomorphism.*
2. *f is a strong monomorphism and an epimorphism.*

Proof. Obviously, only (2) $\Rightarrow$ (1) is of interest. Apply Theorem 3.13 to the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[d, "id_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \square$$

The following is the main property of strong epimorphisms.

Theorem 3.15. *Let C be a category with direct sums. Suppose*

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram in pro-C. *If f' is a strong epimorphism and f is a monomorphism, then there is a unique filler $u:T \to X$, i.e. a morphism u such that $g' = u \circ f'$ and $f \circ u = g$.*

Proof. Since f is a monomorphism, it suffices to prove existence of u. Also, it suffices to prove that $g = f \circ u$. Indeed, $g = f \circ u$ implies $f \circ (u \circ f') = g \circ f' = f \circ g'$, so $u \circ f' = g'$ as f is a monomorphism.

Assume $f:X \to Y$ is a level morphism such that $I(X)$ is cofinite. Let $n(\alpha)$ be the number of predecessors of $\alpha \in I(X)$. By induction on $n(\alpha)$ one can find an increasing function $e:I(X) \to I(X)$ such that for any two morphisms $a, b:Z \to X_{e(\alpha)}$ the equality $f_{e(\alpha)} \circ a = f_{e(\alpha)} \circ b$ implies $p(X)_{\alpha}^{e(\alpha)} \circ a = p(X)_{\alpha}^{e(\alpha)} \circ b$ (see Proposition 2.4).

Since the diagram

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & T \arrow[d, "g"] \\ X \arrow[r, "f"] \arrow[d, "p(X)_\alpha"'] & Y \arrow[d, "p(Y)_\alpha"] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \end{tikzcd}$$

is commutative, one has $h_\alpha : T \to X_\alpha$ so that $f_\alpha \circ h_\alpha = p(Y)_\alpha \circ g$. Define $u_\alpha = p(X)_\alpha^{e(\alpha)} \circ h_{e(\alpha)}$. Suppose $\beta > \alpha$. Notice that $f_{e(\alpha)} \circ p(X)_{e(\alpha)}^{e(\beta)} \circ h_{e(\beta)} = p(Y)_{e(\alpha)}^{e(\beta)} \circ f_{e(\beta)} \circ h_{e(\beta)} = p(Y)_{e(\alpha)}^{e(\beta)} \circ p(Y)_{e(\beta)} \circ g = p(Y)_{e(\alpha)} \circ g$. Also, $f_{e(\alpha)} \circ h_{e(\alpha)} = p(Y)_{e(\alpha)} \circ g$, so $p(X)_\alpha^{e(\alpha)} \circ h_{e(\alpha)} = p(X)_\alpha^{e(\alpha)} \circ p(X)_{e(\alpha)}^{e(\beta)} \circ h_{e(\beta)}$, i.e. $u_\alpha = p(X)_\alpha^\beta \circ u_\beta$. That means $u = \{u_\alpha\}_{\alpha \in I(X)}$ is a morphism from T to X. Since $f_\alpha \circ u_\alpha = f_\alpha \circ p(X)_\alpha^{e(\alpha)} \circ h_{e(\alpha)} = p(Y)_\alpha^{e(\alpha)} \circ f_{e(\alpha)} \circ h_{e(\alpha)} = p(Y)_\alpha^{e(\alpha)} \circ p(Y)_{e(\alpha)} \circ g = p(Y)_\alpha \circ g$ for each $\alpha \in I(X)$, we have $f \circ u = g$. □

Corollary 3.16. *Let $f : X \to Y$ be a morphism in* pro-C. *If C has direct sums, then the following statements are equivalent*:

1. *f is an isomorphism.*
2. *f is a strong epimorphism and a monomorphism.*

Proof. Obviously, only (2) ⇒ (1) is of interest. Apply Theorem 3.15 to the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle id_Y} \\ X & \xrightarrow{f} & Y \end{array} \qquad \square$$

4. Movability

In this section, we introduce a new variant of movability and we discuss the connection of movability to various classes of morphisms of pro-categories.

Definition 4.1. $X \in \mathrm{Ob}(\text{pro-}C)$ is *uniformly movable* if the morphism $\{p(X)_\alpha : X \to X_\alpha\}_{\alpha \in I(X)}$ from X to $\mathrm{Sub}_2(X)$ is a strong epimorphism of pro-(pro-C). In view of Proposition 3.7 it means that for each $\alpha \in I(X)$ there is $\beta > \alpha$ and $r : X_\beta \to X$ such that $r_\alpha = p(X)_\alpha^\beta$, which is the classical definition of uniform movability (see [15, p. 160]).

Here is the connection between uniform movability and strong epimorphisms.

Proposition 4.2. (a) *If there is $P \in \mathrm{Ob}(C)$ and a strong epimorphism $f : P \to X$, then X is uniformly movable.*

(b) *If C is a category with inverse limits and X is uniformly movable, then the projection $p : \varprojlim(X) \to X$ is a strong epimorphism.*

(c) *If C is a category with direct sums and $X \in \mathrm{Ob}(\text{pro-}C)$ is uniformly movable, then there is $P \in \mathrm{Ob}(C)$ and a strong epimorphism $P \to X$.*

Proof. (a) Suppose that $f : P \to X$ is a strong epimorphism and $\alpha \in I(X)$. There is $\beta > \alpha$ and $g : X_\beta \to P$ so that $f_\alpha \circ g = p(X)_\alpha^\beta$. Now, $p(X)_\alpha \circ f \circ g = f_\alpha \circ g = p(X)_\alpha^\beta$ which means that X is uniformly movable.

In (b) and (c) assume that for each $\alpha \in I(X)$ there is $\beta(\alpha) > \alpha$ and $g_\alpha : X_{\beta(\alpha)} \to X$ so that $p(X)_\alpha \circ g_\alpha = p(X)_\alpha^{\beta(\alpha)}$.

(b) g_α factors through $\varprojlim(X)$ so that there is $h_\alpha : X_\beta \to \varprojlim(X)$ with $g_\alpha = p \circ h_\alpha$. Now $(p(X)_\alpha \circ p) \circ h_\alpha = p(X)_\alpha^\beta$ which proves that the level morphism $\{p(X)_\alpha \circ p\}_{\alpha \in I(X)}$ is a strong epimorphism (see Proposition 3.7). That morphism is exactly $p : \varprojlim(X) \to X$.

(c) Let $P = \oplus_{\alpha \in I(X)} X_{\beta(\alpha)}$ and let $f : P \to X$ be induced by g_α, $\alpha \in I(X)$. Given $\alpha \in I(X)$ we have $i_\alpha : X_{\beta(\alpha)} \to P$ so that $f_\alpha \circ i_\alpha = p(X)_\alpha \circ f \circ i_\alpha = p(X)_\alpha \circ g_\alpha = p(X)_\alpha^{\beta(\alpha)}$. That means f is a strong epimorphism. □

Remark 4.3. Consider a uniformly movable pro-group G which is not stable. By Proposition 4.2 there is a strong epimorphism $f : P \to G$ from a group P. That epimorphism cannot have a right inverse as G is not stable.

In [15, Theorem 3, p. 162] it is shown that if Y is uniformly movable and $\varprojlim(f) : \varprojlim(X) \to \varprojlim(Y)$ is an epimorphism of C, then f is an epimorphism of pro-C. We derive that result in part (a) below.

Corollary 4.4. *Suppose C is a category with inverse limits and $f : X \to Y$ is morphism of* pro-C *such that Y is uniformly movable.*

(a) *If $\varprojlim(f)$ is an epimorphism of C, then f is an epimorphism of* pro-C.

(b) *If $\varprojlim(f)$ has a right inverse in C, then f is a strong epimorphism of* pro-C.

(c) *If C is a category with direct sums, f is a monomorphism of* pro-C, *and $\varprojlim(f)$ is an isomorphism of C, then f is an isomorphism of* pro-C.

Proof. (a) (respectively (b)) By Proposition 4.2 the projection morphism $\varprojlim(Y) \to Y$ is a strong epimorphism. Therefore the composition $\varprojlim(X) \to \varprojlim(Y) \to Y$ is an epimorphism (respectively, a strong epimorphism by Corollary 3.12) of pro-C. That composition equals $\varprojlim(X) \to X \xrightarrow{f} Y$, so f is an epimorphism (respectively, a strong epimorphism).

(c) It follows from (b) and Corollary 3.16. □

Notice that, for a category C which does not have inverse limits, the analog of $\varprojlim(f)$ being an isomorphism (see Corollary 4.4) is that $f_* :$ Mor$(P, X) \to$ Mor(P, Y) is a bijection for all $P \in$ Ob(C). Our next results should be viewed in that context.

Corollary 4.5. *Suppose C is a category with direct sums or inverse limits and $Y \in$* Ob(pro-C) *is uniformly movable. A monomorphism $f : X \to Y$ of* pro-C *is an isomorphism if and only if $f_* :$* Mor$(P, X) \to$ Mor(P, Y) *is a surjection for each $P \in$* Ob(C).

Proof. Find a strong epimorphism $g : P \to Y$ such that $P \in$ Ob(C). Factor g as $f \circ h$ for some $h : P \to X$. By Corollary 3.12, f is a strong epimorphism, so Corollary 3.16 implies that f is an isomorphism. □

Theorem 4.6. *Suppose C is a category with direct sums and $f: X \to Y$ is a monomorphism of* pro-C. *If $f_*: \mathrm{Mor}(P,X) \to \mathrm{Mor}(P,Y)$ is a surjection for all $P \in \mathrm{Ob}(C)$, then $f_*: \mathrm{Mor}(T,X) \to \mathrm{Mor}(T,Y)$ is a bijection for all uniformly movable objects T of* pro-C.

Proof. Pick a strong epimorphism $f': P \to T$ such that $P \in \mathrm{Ob}(C)$ (see Proposition 4.2). Suppose $g: T \to Y$ is a morphism and find $g': P \to X$ such that $f \circ g' = g \circ f'$. That means the diagram

$$\begin{array}{ccc} P & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative, so there is $u: T \to X$ such that $f \circ u = g$ by Theorem 3.15. $\square$

Proposition 4.7. *Suppose C is a category and $X \in \mathrm{Ob}(\text{pro-}C)$ is uniformly movable. X is dominated by an object of C if there is a monomorphism $f: X \to P$, where $P \in \mathrm{Ob}(C)$.*

Proof. Pick $g: X_\alpha \to P$ for some $\alpha \in I(X)$, so that $f = g \circ p(X)_\alpha$. Choose $h: X_\beta \to X$ with $p(X)_\alpha \circ h = p(X)^\beta_\alpha$. Let $u = h \circ p(X)_\beta: X \to X$. We plan to show that $u = \mathrm{id}_X$ which means that X is dominated by X_β. It suffices to show that $f \circ u = f \circ \mathrm{id}_X$ as f is a monomorphism. Indeed, $f \circ u = f \circ h \circ p(X)_\beta = g \circ p(X)_\alpha \circ h \circ p(X)_\beta = g \circ p(X)^\beta_\alpha \circ p(X)_\beta = g \circ p(X)_\alpha = f$. $\square$

Remark 4.8. In Proposition 4.16 we will see a pro-group G admitting a monomorphism to a group such that G is not stable. Thus, the assumption of G being uniformly movable is essential.

Proposition 4.9. *Suppose $f: X \to Y$ is a morphism of* pro-C *such that $f_*: \mathrm{Mor}(P,X) \to \mathrm{Mor}(P,Y)$ is a surjection for all $P \in \mathrm{Ob}(C)$. If Y is uniformly movable, then f is a strong epimorphism.*

Proof. Suppose f is a level morphism. Given $\alpha \in I(X)$ find $\beta > \alpha$ and $r: Y_\beta \to Y$ satisfying $r_\alpha = p(Y)^\beta_\alpha$. Lift r to X, i.e. find $h: Y_\beta \to X$ with $f \circ h = r$. Notice that $f_\alpha \circ h_\alpha = r_\alpha = p(Y)^\beta_\alpha$ which means f is a strong epimorphism by Proposition 3.7. $\square$

Remark 4.10. The pro-group G from Proposition 4.16 has the property that the inclusion $0 \to G$ from the trivial group induces epimorphisms $f_*: \mathrm{Mor}(P,0) \to \mathrm{Mor}(P,G)$ for all groups P but $0 \to G$ is not a strong epimorphism. Thus, the assumption of G being uniformly movable is essential.

Definition 4.11. $X \in \mathrm{Ob}(\text{pro-}C)$ is *sequentially movable* if the morphism $\mathrm{Sub}_{\omega_0}(X) \to \mathrm{Sub}_2(X)$ is a strong epimorphism of pro-(pro-C). Alternatively, for any increasing sequence s in $I(X)$ there is $\beta > s(1)$ and a morphism $r : X_\beta \to X_s$ such that $r_1 = p(X)^\beta_{s(1)}$.

Remark 4.12. Notice that if X is sequentially movable, then for any increasing sequence s in $I(X)$ and any $k \geqslant 1$ there is $\beta > s(k)$ and a morphism $r : X_\beta \to X_s$ such that $r_k = p(X)^\beta_{s(k)}$.

Proposition 4.13. *If X is a movable object of* pro-C, *then it is sequentially movable.*

Proof. Clearly, if X is uniformly movable, then it is sequentially movable. Notice that for every sequence s in $I(X)$ there is a sequence $t > s$ such that X_t is movable, hence uniformly movable (see [22] or [15, Theorem 4, p. 163]). Thus, there is $\beta = t(k)$ for some $k > 1$ and $u : X_\beta \to X_t$ with $u_1 = p(X)^\beta_{t(1)}$. Put $r = p(X)^t_s \circ u : X_\beta \to X_s$ to get $r_1 = p(X)^\beta_{s(1)}$. □

Proposition 4.14. *If X has the property that each morphism $p(X)_s$ from X to its subtower X_s factors through a sequentially movable object of* pro-C, *then X is sequentially movable.*

Proof. We need to factor through sequentially movable objects whose index set is cofinite. □

Claim. Every sequentially movable object Z is isomorphic to a sequentially movable object Z' such that $I(Z')$ is cofinite.

Proof of Claim. We will employ the standard reindexing trick as in the proof of Theorem 3 [15, p. 12]. Define $I(Z')$ to be the set of all finite subsets σ of $I(Z)$ which have a maximum $\max(\sigma)$ and declare $\sigma \leqslant \tau$ if $\sigma \subset \tau$. Z'_σ is defined as $Z_{\max(\sigma)}$ and $p(Z')^\tau_\sigma := p(Z)^{\max(\tau)}_{\max(\sigma)}$. Given an increasing sequence s in $I(Z')$ we define $t(n) = \max(s(n))$. There is $\gamma > t(1)$ and $u : Z_\gamma \to Z_t$ such that $u_1 = p(Z)^\gamma_{t(1)}$. Setting $\beta = s(1) \cup \{\gamma\}$ and interpreting u as $r : Z'_\beta \to Z'_s$ one gets $r_1 = p(Z')^\beta_{s(1)}$. Thus Z' is sequentially movable. Notice that projections from Z' to Z_α, where α is interpreted as a one-point set, form a morphism from Z' to Z which is an isomorphism. □

Suppose s is an increasing sequence in $I(X)$. Factor $p(X)_s : X \to X_s$ as $h \circ g$, where $g : X \to Z$, $h : Z \to X_s$, Z is sequentially movable, and $I(Z)$ is cofinite. Find a sequence t in $I(Z)$ and a level morphism $f : Z_t \to X_s$ such that $h = f \circ p(Z)_t$. For each $\alpha \in I(Z)$ let $n(\alpha)$ be the number of predecessors of α. By induction on $n(\alpha)$ we can construct an increasing function $i : I(Z) \to I(X)$ and representatives $g_\alpha : X_{i(\alpha)} \to Z_\alpha$ of $p(Z)_\alpha \circ g$ such that $\{g_\alpha\}_{\alpha \in I(Z)}$ induces a level morphism, i.e. $p(Z)^\beta_\alpha \circ g_\beta = g_\alpha \circ p(X)^{i(\beta)}_{i(\alpha)}$ for all $\alpha < \beta$. Pick $\gamma \in I(Z)$, $\gamma > t(1)$, and $u : Z_\gamma \to Z_t$ such that $u_1 = p(Z)^\gamma_{t(1)}$. Set $\beta = i(\gamma)$ and $r = f \circ u \circ g_\gamma$. Now $p(X_s)_1 \circ r = p(X_s)_1 \circ f \circ u \circ g_\gamma = f_1 \circ p(Z_t)_1 \circ u \circ g_\gamma = f_1 \circ$

$p(Z)^{\gamma}_{t(1)} \circ g_{\gamma} = f_1 \circ g_{t(1)} \circ p(X)^{\beta}_{t(1)} = p(X)^{i(t(1))}_{s(1)} \circ p(X)^{\beta}_{t(1)} = p(X)^{\beta}_{s(1)}$ and X is sequentially movable. □

Corollary 4.15. *If X is dominated by a sequentially movable object of* pro-C, *then it is sequentially movable.*

Let us show that sequential movability is a more general concept than movability.

Proposition 4.16. *There is a sequentially movable* pro-*group which is not movable.*

Proof. Let $I(X)$ be the set of all ordinals smaller than the first uncountable ordinal. For each $\alpha \in I(X)$ let $X_{\alpha} = \oplus_{\gamma \in I(X)} G^{\alpha}_{\gamma}$, where $G^{\alpha}_{\gamma} = 0$ if $\gamma < \alpha$ and G^{α}_{γ} is the group of natural numbers Z otherwise. $p(X)^{\beta}_{\alpha} : X_{\beta} \to X_{\alpha}$ is the natural inclusion. Any increasing sequence s in $I(X)$ has an upper bound β. Therefore any morphism from X to a tower factors through a group. By Proposition 4.14 that proves the sequential movability of X.

If X were movable, then for each α there would be $\beta > \alpha$ with $\mathrm{im}(p(X)^{\beta}_{\alpha})$ contained in each $\mathrm{im}(p(X)^{\gamma}_{\alpha})$, $\gamma > \beta$. However, that implies $\mathrm{im}(p(X)^{\beta}_{\alpha})$=0, a contradiction. Notice that Proposition 5.1 generalizes the above argument. □

Remark 4.17. The same argument as in Proposition 4.16 shows that the system of subtowers $\mathrm{Sub}_{\omega_0}(X)$ of X is always sequentially movable. That is because every sequence in $\mathrm{Sub}_{\omega_0}(X)$ has an upper bound.

Remark 4.18. Consider the inclusion $X \to X_0$ as in Proposition 4.16. It is a strong monomorphism of pro-*Gr* which does not have a left inverse as X is not stable.

Proposition 4.19. *If $f : X \to Y$ is a strong epimorphism and X is sequentially movable, then Y is sequentially movable.*

Proof. Without loss of generality assume that f is a level morphism. Given a sequence s in $I(Y)$ find $\gamma \in I(X)$, $\gamma > s(1)$, and $u : X_{\gamma} \to X_s$ such that $u_1 = p(X)^{\gamma}_{s(1)}$. Find $\beta > \gamma$ and $g : Y_{\beta} \to X_{\gamma}$ such that $f_{\gamma} \circ g = p(Y)^{\beta}_{\gamma}$. Set $r = f_s \circ u \circ g : Y_{\beta} \to Y_s$. Notice that $r_1 = f_1 \circ u_1 \circ g = f_1 \circ p(X)^{\gamma}_{s(1)} \circ g = p(Y)^{\gamma}_{s(1)} \circ f_{\gamma} \circ g = p(Y)^{\gamma}_{s(1)} \circ p(Y)^{\beta}_{\gamma} = p(Y)^{\beta}_{s(1)}$. □

Proposition 4.20. *Suppose $f : X \to Y$ is a level morphism of* pro-C *such that Y is sequentially movable. Let Σ be the set of sequences s in $I(X)$ such that $(f_s)_* :$ $\mathrm{Mor}(P, X_s) \to \mathrm{Mor}(P, Y_s)$ is a surjection for all $P \in \mathrm{Ob}(C)$. If Σ is cofinal in the set of all sequences in $I(X)$, then f is a strong epimorphism.*

Proof. Given $\alpha \in I(X)$ find a sequence s in $I(X)$ such that $s(1) > \alpha$ and $(f_s)_* :$ $\mathrm{Mor}(P, X_s) \to \mathrm{Mor}(P, Y_s)$ is a surjection for all $P \in \mathrm{Ob}(C)$. Pick $\beta > \alpha$ and $r : Y_{\beta} \to Y_s$ satisfying $r_1 = p(Y)^{\beta}_{s(1)}$. Lift r to X_s, i.e. find $r' : Y_{\beta} \to X_s$ with $f_s \circ r' = r$. Set $g : Y_{\beta} \to X_{\alpha}$

to be $p(X)^{s(1)}_{\alpha} \circ p(X_s)_1 \circ r'$ and notice that $f_{\alpha} \circ g = f_{\alpha} \circ p(X)^{s(1)}_{\alpha} \circ p(X_s)_1 \circ r' = p(Y)^{s(1)}_{\alpha} \circ p(Y_s)_1 \circ f_s \circ r' = p(Y)^{s(1)}_{\alpha} \circ p(Y_s)_1 \circ r = p(Y)^{\beta}_{\alpha}$. □

Theorem 4.21. *Suppose C is a category with direct sums and $f : X \to Y$ is a monomorphism of* tow(C). *If $f_* :$ Mor$(P,X) \to$ Mor(P,Y) is a surjection for all $P \in$ Ob(C), then $f_* :$ Mor$(Z,X) \to$ Mor(Z,Y) is a bijection for all sequentially movable objects Z of* pro-C.

Proof. *Special Case*. f is a level morphism induced by $\{f_n : X_n \to Y_n\}_{n\in N}$ such that for any two morphisms $a, b : P \to X_{n+1}$ of C the equality $f_{n+1} \circ a = f_{n+1} \circ b$ implies $p(X)^{n+1}_n \circ a = p(X)^{n+1}_n \circ b$.

Suppose $g : Z \to Y$ is a morphism of pro-C and Z is sequentially movable. By 2.10 there is an increasing sequence s in $I(Z)$ and a level morphism $h : Z_s \to Y$ satisfying $g = h \circ p(Z)_s$. By induction on n we can construct an increasing sequence t in $I(Z)$, $t > s$, and morphisms $r(n) : Z_{t(n)} \to Z_s$, $n \geqslant 1$, so that $r(n)_n = p(Z)^{t(n)}_{s(n)}$. Lift each $h \circ r(n)$ to X, i.e. find $h(n) : Z_{t(n)} \to X$ so that $f \circ h(n) = h \circ r(n)$. Define $u(n) = t(n+1)$, $n \geqslant 1$. We plan to show that $k_n = p(X)^{n+1}_n \circ h(n+1)_{n+1}$, $n \geqslant 1$, induce a level morphism k from Z_u to X such that $f \circ (k \circ p(Z)_u) = g$.

We need to prove $p(X)^{n+1}_n \circ k_{n+1} = k_n \circ p(Z)^{t(n+2)}_{t(n+1)}$ for all n. If we show $f_{n+1} \circ k_{n+1} = f_{n+1} \circ h(n+1)_{n+1} \circ p(Z)^{t(n+2)}_{t(n+1)}$, then it implies $p(X)^{n+1}_n \circ k_{n+1} = p(X)^{n+1}_n \circ h(n+1)_{n+1} \circ p(Z)^{t(n+2)}_{t(n+1)} = k_n \circ p(Z)^{t(n+2)}_{t(n+1)}$, i.e. what we need.

$f_{n+1} \circ h(n+1)_{n+1} \circ p(Z)^{t(n+2)}_{t(n+1)} = h_{n+1} \circ r(n+1)_{n+1} \circ p(Z)^{t(n+2)}_{t(n+1)} = h_{n+1} \circ p(Z)^{t(n+1)}_{s(n+1)} \circ p(Z)^{t(n+2)}_{t(n+1)} = h_{n+1} \circ p(Z)^{t(n+2)}_{s(n+1)}$. Also, $f_{n+1} \circ k_{n+1} = f_{n+1} \circ p(X)^{n+2}_{n+1} \circ h(n+2)_{n+2} = p(Y)^{n+2}_{n+1} \circ f_{n+2} \circ h(n+2)_{n+2} = p(Y)^{n+2}_{n+1} \circ h_{n+2} \circ r(n+2)_{n+2} = p(Y)^{n+2}_{n+1} \circ h_{n+2} \circ p(Z)^{t(n+2)}_{s(n+2)} = h_{n+1} \circ p(Z)^{s(n+2)}_{s(n+1)} \circ p(Z)^{t(n+2)}_{s(n+2)} = h_{n+1} \circ p(Z)^{t(n+2)}_{s(n+1)}$. Thus $f_{n+1} \circ k_{n+1} = f_{n+1} \circ h(n+1)_{n+1} \circ p(Z)^{t(n+2)}_{t(n+1)}$. Also, we established $f_n \circ k_n = h_n \circ p(Z)^{t(n+1)}_{s(n)}$ which means $f \circ k = h \circ p(Z)^u_s$. Composing with $p(Z)_u$ gives $f \circ k \circ p(Z)_u = h \circ p(Z)_s = g$.

General Case: Using Propositions 2.10 and 2.4 we can find an increasing sequence $s : N \to N$ such that $f' = p(X)_s \circ f : X_s \to Y$ is a level morphism and for any two morphisms $a, b : P \to X_{s(n+1)}$ of C the equality $f'_{n+1} \circ a = f'_{n+1} \circ b$ implies $p(X_s)^{n+1}_n \circ a = p(X_s)^{n+1}_n \circ b$. Notice that $p(X)_s : X \to X_s$ is an isomorphism as $s(N)$ is cofinal in N. By the Special Case $f'_* :$ Mor$(Z,X_s) \to$ Mor(Z,Y) is a bijection for all sequentially movable objects Z of pro-C. As $p(X)_s$ is an isomorphism, $f_* :$ Mor$(Z,X) \to$ Mor(Z,Y) is a bijection for all sequentially movable objects Z of pro-C. □

Remark 4.22. The above result is not valid for arbitrary monomorphisms $f : X \to Y$. The pro-group G in Proposition 4.16 is sequentially movable, the trivial morphism $f : 0 \to G$ has the property that $f_* :$ Mor$(P,0) \to$ Mor(P,G) is a bijection for all groups P but $f_* :$ Mor$(G,0) \to$ Mor(G,G) is not a surjection.

5. Stability in pro-categories

The next two results will be useful for applications to pro-groups.

Proposition 5.1. *If X is a* pro-*object such that each $p(X)^{\beta}_{\alpha}$ is a monomorphism of C, then the following conditions are equivalent*:

(1) *X is stable.*
(2) *X is movable.*
(3) *There is $\alpha \in I(X)$ such that each $p(X)^{\beta}_{\alpha}$, $\beta > \alpha$, is an isomorphism of C.*

Proof. (1) ⇒ (2) follows from the fact that movability is preserved by isomorphisms (see [15, Theorem 1, p. 159]).

(2) ⇒ (3). Let $\alpha \in I(X)$ and pick $\beta > \alpha$ such that for each $\gamma > \beta$ there is $r_{\gamma} : X_{\beta} \to X_{\gamma}$ with $p(X)^{\gamma}_{\alpha} \circ r_{\gamma} = p(X)^{\beta}_{\alpha}$. Now $p(X)^{\beta}_{\alpha} \circ (p(X)^{\gamma}_{\beta} \circ r_{\gamma}) = p(X)^{\beta}_{\alpha} \circ \mathrm{id}(X_{\beta})$, so $p(X)^{\gamma}_{\beta} \circ r_{\gamma} = \mathrm{id}(X_{\beta})$. That means $p(X)^{\gamma}_{\beta}$ has a left inverse and must be an isomorphism by Proposition 1.2.

(3) ⇒ (1): Notice that $p(X)_{\alpha} : X \to X_{\alpha}$ is an isomorphism. □

Proposition 5.2 (Dydak and Ruiz de Portal [9, Corollary 2.12]). *Suppose X is a* pro-*object such that each $p(X)^{\beta}_{\alpha}$ is an epimorphism of C. X is stable if and only if there is $\alpha \in I(X)$ such that $p(X)^{\beta}_{\alpha}$ is an isomorphism of C for all $\beta > \alpha$.*

In [9] the authors discussed the question of objects of C having stable images (respectively, stable subobjects) in pro-C. In this section we deal with analogous question of objects having stable strong images (respectively, stable strong subobjects). We will use repeatedly the following result.

Corollary 5.3 (Dydak and Ruiz de Portal [9, 3.6]). *Suppose C is a balanced category with epimorphic images. If* pro-*C is balanced, then for any epimorphism* (*respectively*, *monomorphism*) *$f : X \to Y$ of* pro-*C there exists a level morphism $f' : X' \to Y'$ and isomorphisms $i : X \to X'$, $j : Y' \to Y$ such that $f = j \circ f' \circ i$, $I(X')$ is a cofinite directed set, and f'_{α} is an epimorphism* (*respectively*, *monomorphism*) *of C for each $\alpha \in I(Y')$. Moreover, the bonding morphisms of X'* (*respectively*, *Y'*) *are chosen from the set of bonding morphisms of X* (*respectively*, *Y*).

Recall that C is a balanced category with epimorphic images if every morphism f of C has a unique, up to isomorphism, decomposition $f = f'' \circ f'$ such that f' is an epimorphism of C and f'' is a monomorphism of C.

Definition 5.4. Let C be a category. $Y \in \mathrm{Ob}(\text{pro-}C)$ is called a *strong image* (respectively, *a strong subobject*) of an object X of pro-C provided there is a strong epimorphism (respectively, strong monomorphism) $f : X \to Y$ (respectively, $f : Y \to X$) of pro-C.

Definition 5.5. Let C be a category. An object P of C has *stable strong images* (respectively, *stable strong subobjects*) if any strong image (respectively, strong subobject) $X \in \mathrm{Ob}(\text{pro-}C)$ of P is stable.

Theorem 5.6. *Let R be a principal ideal domain. If P is a finitely generated R-module, then it has stable strong images (respectively, stable strong subobjects) in the* pro-*category* pro-M_R *of the category M_R of R-modules.*

Proof. Suppose $f : X \to P$ is a strong monomorphism. Since M_R is a balanced category with epimorphic images and pro-M_R is balanced, Corollary 5.3 allows us to reduce the proof to the case where f is a level morphism, and each f_α is a monomorphism. In particular, as $f_\beta = f_\alpha \circ p(X)^\beta_\alpha$, each $p(X)^\beta_\alpha$ is a monomorphism. Subsequently, we may simply identify all X_α with submodules of P so that all $p(X)^\beta_\alpha$ are inclusion-induced. Now it suffices to show that there is $\alpha \in I(X)$ such that $X_\beta = X_\alpha$ for all $\beta > \alpha$.

Special Case: P is a torsion R-module. In this case P satisfies the descending chain condition on submodules in view of [13, Theorem 1.5, p. 373], so X is stable.

Notice that one has the functor Tor $: M_R \to M_R$ such that $\mathrm{Tor}(Q)$ is the torsion part of an R-module Q. That functor can be extended to Tor : pro-$M_R \to$ pro-M_R. Therefore $\mathrm{Tor}(X)$ is a strong subobject of $\mathrm{Tor}(P)$ and must be stable by Special Case. Without loss of generality we may assume $\mathrm{Tor}(X_\alpha) = \mathrm{Tor}(X_\beta)$ for all $\beta > \alpha$. Also, as the rank of the free part of X_α is at most the rank of the free part of P, we may assume that the ranks of all free parts of modules X_α are equal to a fixed natural number m. Suppose X is not stable. By Proposition 2.3 there is a triple $\gamma > \beta > \alpha$ of elements of $I(X)$ such that for some morphism $r : P \to X_\beta$ one has $r|X_\gamma$ is the inclusion and $X_\gamma \neq X_\beta \neq X_\alpha$. Notice that $r|T$ is the inclusion, where T is the torsion submodule of X_α, so one can put $Q = P/T$, $Y_\delta = X_\delta/T$ for all $\delta \in I(X)$, and get $s : Q \to Y_\beta$ such that $s|Y_\gamma$ is the inclusion and $Y_\gamma \neq Y_\beta \neq Y_\alpha$ are all free R-modules of the same rank. Let $M = \{x \in Y_\alpha \mid s(x) = x\}$. It is a proper submodule of Y_α containing Y_γ and contained in Y_β, so it has the same rank as Y_α and cannot be a direct summand of Y_α. Therefore Y_α/M is not torsion-free and there is $u \in Y_\alpha \setminus M$ with $q \cdot u \in M$ for some $q \in R \setminus \{0\}$. Thus $q \cdot (s(u) - u) = 0$ implying $s(u) - u = 0$ and $u \in M$, a contradiction.

Suppose $f : P \to X$ is a strong epimorphism. Since M_R is a balanced category with epimorphic images and pro-M_R is balanced, Corollary 5.3 allows us to reduce the proof to the case where f is a level morphism and each f_α is an epimorphism. Since $f_\alpha = p(X)^\beta_\alpha \circ f_\beta$, each $p(X)^\beta_\alpha$ is an epimorphism. Now it suffices to show that there is $\alpha \in I(X)$ such that $\ker(f_\beta) = \ker(f_\alpha)$ for all $\beta > \alpha$.

Special Case. P is a torsion R-module. In this case P satisfies the descending chain condition on submodules in view of [13, Theorem 1.5, p. 373], so X is stable.

Notice that $\mathrm{Tor}(X)$ is a strong image of $\mathrm{Tor}(P)$ and must be stable by Case 1. Without loss of generality we may assume $p(X)^\beta_\alpha|\mathrm{Tor}(X_\beta)$ sends $\mathrm{Tor}(X_\beta)$ isomorphically onto $\mathrm{Tor}(X_\alpha)$ for all $\beta > \alpha$. Also, as the rank of the free part of X_α is at most the rank of the free part of P, we may assume that the ranks of all free parts of modules X_α are equal to a fixed natural number m. Therefore $\ker(p(X)^\beta_\alpha)$ must be a torsion module for all $\beta > \alpha$. Since $p(X)^\beta_\alpha|\mathrm{Tor}(X_\beta)$ sends $\mathrm{Tor}(X_\beta)$ isomorphically onto $\mathrm{Tor}(X_\alpha)$ for all $\beta > \alpha$, $p(X)^\beta_\alpha$ must be an isomorphism. $\square$

The same proof as in Theorem 5.6 yields the following. One cannot derive Corollary 5.7 formally from Theorem 5.6 as the category of groups is larger than the category of Z-modules (i.e., the category of Abelian groups).

Corollary 5.7. *If P is a finitely generated Abelian group, then it has stable strong images (respectively, stable strong subobjects) in the category of* pro-*groups.*

The remainder of this section is devoted to describing another class of Abelian groups whose members have stable strong images (respectively, stable strong subobjects) in the category of pro-groups.

Definition 5.8 (Fuchs [11, p. 29]). Let S be a subset of an Abelian group G. S is called *linearly independent*, or briefly, *independent*, if any relation

$$n_1 \cdot a_1 + \cdots + n_k \cdot a_k = 0$$

implies $n_i \cdot a_i = 0$ for all i.

Definition 5.9 (Fuchs [11, p. 31]). Let G be an Abelian group G. The *rank* $r(G)$ of G is the cardinality of a maximal independent subset of G whose elements have orders being prime or infinity.

Theorem 5.10. *Let G be a divisible Abelian group. G has stable strong images (respectively, stable strong subobjects) in the category of pro-groups if and only if its rank is finite.*

Proof. Suppose the rank of G is infinite. By Theorem 19.1 of [11, p. 64] one can express G as the direct sum of an infinite sequence G_i, $i > 0$, of some of its non-trivial subgroups. Let F_n be the direct sum of G_i, $i \leqslant n$. Notice that one has a tower F bonded by projections such that the inclusion $F \to G$ is a strong monomorphism and F is not stable. Similarly, let H_n be the direct sum of G_i, $i \geqslant n$. Notice that one has a tower H bonded by inclusions such that there is a strong epimorphism $G \to H$ and H is not stable. $\square$

Assume the rank r of G is finite.

Claim. *Given a descending chain G_i of divisible subgroups of G there is k such that $G_{i+1} = G_i$ for $i \geqslant k$.*

Proof. It suffices to show that one cannot have a descending sequence $G_0 = G \supset G_1 \supset \cdots \supset G_{r+1} \neq 0$ such that each G_{i+1} is a proper divisible subgroup of G_i for $i = 0, \ldots, r$. By Theorem 18.1 of [11, p. 62] each G_{i+1} is a direct summand of G_i. Therefore, starting from a maximal independent subset S_{r+1} of G_{r+1} (consisting of elements whose orders are prime or infinity) one can increase it to a maximal independent subset S_r of G_r (consisting of elements whose orders are prime or infinity) and so on. In the end we will get a maximal independent subset S_0 of G (consisting of elements whose orders are prime or infinity) whose cardinality is larger than r, a contradiction.

Suppose $f:X \to G$ is a strong monomorphism. Since *Gr* is a balanced category with epimorphic images and pro-*Gr* is balanced, Corollary 5.3 allows us to reduce the proof to the case where each $p(X)_\alpha^\beta$ is a monomorphism, f is a level morphism, and each f_α is a monomorphism. Subsequently, we may simply identify all X_α with submodules of P so that all $p(X)_\alpha^\beta$ are inclusion-induced. Now it suffices to show that there is $\alpha \in I(X)$ such that $X_\beta = X_\alpha$ for all $\beta > \alpha$. Suppose X is not stable. We may reduce the general case to the one of X being a tower such that for each n there is a homomorphism $r_n : G \to X_n$ so that $r_n|X_{n+1} = id$ and X_{n+1} is a proper subgroup of X_n. Let $G_n = r_n(G)$. It is a divisible subgroup of G and $X_{n+1} \subset G_n \subset X_n$. There is k such that $G_{n+1} = G_n$ for $n \geqslant k$ implying $X_{n+1} = X_n$ for $n > k$, a contradiction.

Suppose $f:G \to X$ is a strong epimorphism. Since *Gr* is a balanced category with epimorphic images and pro-*Gr* is balanced, Corollary 5.3 allows us to reduce the proof to the case where each $p(X)_\alpha^\beta$ is an epimorphism, f is a level morphism, and each f_α is an epimorphism. Now it suffices to show that there is $\alpha \in I(X)$ such that $\ker(f_\beta) = \ker(f_\alpha)$ for all $\beta > \alpha$. Suppose X is not stable. Again, we can reduce the general case to that of X being a tower, $\ker(f_{n+1})$ being a proper subgroup of $\ker(f_n)$ for each n, and the equality $f_n \circ r_n = p(X)_n^{n+1}$ for some homomorphism $r_n : X_{n+1} \to G$. Moreover, we may assume that every divisible subgroup of G contained in all $\ker(f_n)$, $n \geqslant 1$, is trivial. Indeed, one can consider a maximal divisible subgroup D of the intersection of all $\ker(f_n)$, $n \geqslant 1$. D is a direct summand of G, so replacing G by G/D does the trick. Let H_n be the subgroup of G generated by elements $x - r_m f_{m+1}(x)$, where $x \in G$ and $m \geqslant n$. Notice that H_n is divisible and $H_{n+1} \subset H_n$. Let us show $H_n \subset \ker(f_n)$. Indeed, $f_n(r_m f_{m+1}) = (p(X)_n^m \circ f_m) \circ r_m \circ f_{m+1} = p(X)_n^m \circ p(X)_m^{m+1} \circ f_{m+1} = f_n$ for $m \geqslant n$, so $f_n(x - r_m f_{m+1}(x)) = 0$. Pick $k \geqslant 1$ such that $H_m = H_n$ for $m, n \geqslant k$. That means $H_n = 0$ for $n \geqslant k$ proving that $f : G \to X$ is an isomorphism. Thus X is stable, a contradiction. □

6. Bimorphisms in pro-categories

The purpose of this section is to relate bimorphisms of pro-C to bimorphisms of tow(C) for categories C with direct sums and weak push-outs.

Lemma 6.1. *Suppose C is a category with direct sums and weak push-outs. If $f:X \to Y$ is a level morphism of* pro-C *which is a bimorphism of* pro-C, *then the set of sequences s in $I(X)$ such that $f_s : X_s \to Y_s$ is a bimorphism of* tow(C) *is cofinal among all sequences in $I(X)$.*

Proof. Let $g : I(X) \times I(X) \to I(X)$ be a function such that $g(\alpha, \beta) > \alpha, \beta$. Using Propositions 2.4 and 2.7 there are functions $m : I(X) \to I(X)$ and $e : I(X) \to I(X)$ with the following properties:

(1) $m(\alpha) > \alpha$ and for any two morphisms $a, b : P \to X_{m(\alpha)}$ the equality $f_{m(\alpha)} \circ a = f_{m(\alpha)} \circ b$ implies $p(X)_\alpha^{m(\alpha)} \circ a = p(X)_\alpha^{m(\alpha)} \circ b$.

(2) $e(\alpha) > \alpha$ and for any two morphisms $a, b : Y_\alpha \to P$ the equality $a \circ f_\alpha = b \circ f_\alpha$ implies $a \circ p(Y)_\alpha^{e(\alpha)} = b \circ p(Y)_\alpha^{e(\alpha)}$.

Given any sequence t in $I(X)$ define $s(1) = g(m(t(1)), e(t(1)))$ and, inductively, $s(n+1) = g(g(m(t(n)), e(t(n))), s(n))$. Using Propositions 2.4 and 2.7 it is easy to check that f_s is a bimorphism of pro-C. □

Theorem 6.2. *If C is a category with direct sums and weak push-outs, then the following conditions are equivalent*:

(1) tow(C) *is balanced.*
(2) pro-C *is balanced.*

Proof. (1) ⇒ (2). Suppose $f : X \to Y$ is a level morphism of pro-C which is a bimorphism of pro-C. We can find a cofinal subset Σ of the set of increasing sequences in $I(X)$ such that $f_s : X_s \to Y_s$ is a bimorphism of pro-C for each $s \in \Sigma$. Now, each f_s is an isomorphism, so f is an isomorphism by Proposition 2.13.

(2) ⇒ (1). This amounts to showing that any bimorphism of tow(C) is also a bimorphism of pro-C. That was done in Corollary 2.12. □

7. Weak equivalences in pro-homotopy

Recall that *a weak equivalence* in pro-H_0 is a morphism $f : X \to Y$ such that pro-$\pi_n(f)$ is an isomorphism for all n. Also, *the deformation dimension* $\dim_{\mathrm{def}}(X)$ of X is the smallest number n such that for any $\alpha \in I(X)$ there is $\beta > \alpha$ with $p(X)_\alpha^\beta$ having a representative with image contained in the n-skeleton of X_α (see [5]).

The purpose of this section is to generalize some versions of the Whitehead Theorem in pro-homotopy (see [5,15]).

Lemma 7.1. Any weak equivalence $g : X \to Y$ of tow(H_0) has the property that $g_* : \mathrm{Mor}(P, X) \to \mathrm{Mor}(P, Y)$ is surjective for all CW complexes P.

Proof. Notice that every object of tow(H_0) is equivalent to a tower of spaces homotopically equivalent to pointed connected CW complexes so that bonding maps are Hurewicz fibrations. (see [5, Theorem 5.2 and its proof]). Let us assume X and Y are towers in the category of spaces homotopically equivalent to pointed connected CW complexes as objects and Hurewicz fibrations as morphisms. Using Proposition 2.10 we can reduce the proof to the case of f being a level morphism. Moreover, as $p(Y)_n^m$ are Hurewicz fibrations, we may assume that f_n are actually maps (as opposed to homotopy classes of maps) so that $p(Y)_n^m \circ f_m = f_n \circ p(X)_n^m$ for $m > n$. Let $\bar{X}$ (respectively, $\bar{Y}$) be the inverse limit of X (respectively, Y) and let $\bar{f} : \bar{X} \to \bar{Y}$ be the map induced by f. Notice that $\mathrm{Mor}(P, \bar{X}) \to \mathrm{Mor}(P, X)$ is an epimorphism for all pointed connected CW complexes P, and the same statement holds for Y. It follows from the fact that bonding maps are Hurewicz fibrations (see [5, Theorem 5.2 and its proof]). Therefore, it suffices to show that $\bar{f}$ is a weak homotopy equivalence. By Bousfield and Kan [3, p. 254] one has the following short exact sequence:

$$0 \to \varprojlim{}^1 \pi_{i+1}(X) \to \pi_i(\bar{X}) \to \varprojlim \pi_i(X) \to 0.$$

Since the same sequence holds for Y, the Five Lemma implies that $\bar{f}$ is a weak homotopy equivalence. □

Theorem 7.2. *Suppose $f:X \to Y$ is a weak equivalence of* pro-H_0. *If Y is sequentially movable, then f is a strong epimorphism of* pro-H_0.

Proof. Assume that f is a level morphism of pro-H_0. As in [5] or using the same technique as in the proof of 6.1 (see also [15, p. 160]) one notices that the set of sequences s in $I(X)$ such that f_s is a weak equivalence is cofinal among all sequences in $I(X)$. By Proposition 4.20 and Lemma 7.1, f is a strong epimorphism of pro-H_0. □

Theorem 7.3. *Suppose $f:X \to Y$ is a weak equivalence of* pro-H_0. *f is an isomorphism of* pro-H_0 *in the following two cases*:

1. $\dim_{\mathrm{def}}(X)$ *is finite and Y is sequentially movable.*
2. $\dim_{\mathrm{def}}(Y)$ *is finite and X is sequentially movable.*

Proof. In case of (1) f is a strong epimorphism of pro-H_0 and it is shown in [5] that $\dim_{\mathrm{def}}(Y) \leqslant \dim_{\mathrm{def}}(X)$ in such a case. Therefore, both X and Y are of finite deformation dimension and f is an isomorphism of pro-H_0 by, Dydak [5] (see also [15, Theorem 3 on p. 149]).

In case of (2) there is a right inverse $g:Y \to X$ as shown in [5] (see [15, Theorem 4, pp. 149–150]). By case (1) g is an isomorphism, so f is an isomorphism of pro-H_0 as well. □

8. Bimorphisms in pro-homotopy

In this section we give partial answers to the following question.

Problem 8.1. If $f:X \to Y$ is a bimorphism in pro-H_0, is f an isomorphism?

Notice that H_0 has direct sums in the form of the wedge of CW complexes. Also, H_0 has weak push-outs in the form of the union of mapping cylinders.

Proposition 8.2. *Suppose $f:X \to Y$ is a level morphism of* pro-H_0 *such that for every $\alpha \in I(X)$ there is $\beta > \alpha$ with the property that for any morphisms $a,b:\Sigma(P) \to X_\beta$ of H_0, the equality $f_\beta \circ a = f_\beta \circ b$ implies $p(X)^\beta_\alpha \circ a = p(X)^\beta_\alpha \circ b$. If f is an epimorphism of* pro-H_0, *then it is a weak equivalence of* pro-H_0.

Proof. In case of f being a morphism between towers it follows from Theorem 2.10 of [8]. Indeed, the above condition implies that $\pi_k(f)$ is a monomorphism for all $k \geqslant 1$ and 2.10 of [8] says that any epimorphism of tow(H_0) is a weak equivalence in such a case. In the general case one reduces the problem to $f_s : X_s \to Y_s$, where s is a sequence in $I(X)$ so that f_s satisfies the assumptions of this proposition. The set of such s is cofinal among all sequences in $I(X)$. Since each f_s is a weak equivalence, so is f. □

Theorem 8.3. *If $f : X \to Y$ is a bimorphism in* pro-H_0, *then it is a weak equivalence.*

Proof. Assume f is a level morphism. Use Proposition 8.2 and Theorem 2.4. □

Theorem 8.4. *Suppose $f : X \to Y$ is a bimorphism in* pro-H_0. *Then f is an isomorphism if one of the following conditions is satisfied*:

(i) *Y is sequentially movable*,
(ii) $\dim_{\mathrm{def}}(Y)$ *is finite*.

Proof. By Theorem 8.3, f is a weak equivalence. In case of (i) f is a strong epimorphism by 7.2. Therefore, by Corollary 3.16, it is an isomorphism.

In case of (ii) f has a right inverse (see [5]), so it must be an isomorphism. □

Theorem 8.5. *If $f : X \to Y$ is a bimorphism in* pro-H_0, *then $f_* :$ Mor$(Z, X) \to$ Mor(Z, Y) *is a bijection for all sequentially movable objects Z of* pro-H_0.

Proof. Using Proposition 2.13 and as in Lemma 6.1 one can reduce it to f being a level morphism of tow(H_0). Since f is a weak equivalence, then $f_* :$ Mor$(P, X) \to$ Mor(P, Y) is a surjection for all CW complexes P (see Lemma 7.1). By Theorem 4.21, $f_* :$ Mor$(Z, X) \to$ Mor(Z, Y) is a bijection for all sequentially movable objects Z of pro-H_0. □

Recall that $\pi_*(P)$ is the group of homotopy classes of maps from $\bigvee_{i=1}^{\infty} S^i$ to P. As a consequence of the above theorem one gets, in view of Corollary 4.4, the following.

Corollary 8.6. *If $f : X \to Y$ is a bimorphism in* pro-H_0 *and* pro-$\pi_*(Y)$ *is uniformly movable then $f_* :$* pro-$\pi_*(X) \to$ pro-$\pi_*(Y)$ *is an isomorphism.*

Proof. Notice that f_* is a monomorphism for any bimorphism $f : X \to Y$ by part (b) of Proposition 2.4 applied to $P = \bigvee_{i=1}^{\infty} S^i$.

Let $g = \varprojlim(f_*)$. Applying Theorem 8.5 to $Z = \bigvee_{i=1}^{\infty} S^i$ one gets that g is an isomorphism as $\varprojlim(\text{pro-}\pi_*(X))$ equals $\mathrm{Mor}_{\text{pro-}H_0}(Z, X)$ and $\varprojlim(\text{pro-}\pi_*(Y)) = \mathrm{Mor}_{\text{pro-}H_0}(Z, Y)$. Now, Part (c) of Corollary 4.4 says that f_* is an isomorphism if pro-$\pi_*(Y)$ is uniformly movable. □

9. Bimorphisms in the shape category

Let HT_0 be the homotopy category of pointed connected topological spaces. A *shape system* of $X \in \mathrm{Ob}(HT_0)$ is an object K of pro-H_0 such that for some morphism $f : X \to K$ of pro-HT_0 the induced function $f^* : \mathrm{Mor}(K, L) \to \mathrm{Mor}(X, L)$ is a bijection for all $L \in \mathrm{Ob}(H_0)$. There is (see [15]) a *shape category* Sh and the shape functor S: Sh $\to$ pro-H_0 such that $S(X)$ is the shape system for each pointed connected topological space X and S establishes a bijection between $\mathrm{Mor}_{\mathrm{Sh}}(X, Y)$ and $\mathrm{Mor}_{\text{pro-}H_0}(S(X), S(Y))$. In this sense one can identify X with $S(X)$ and consider Sh to be the full subcategory of pro-H_0 whose objects are shape systems of pointed connected topological spaces. This is the approach we take in this section.

Definition 9.1. Given a directed set A and pointed CW-complexes $\{P_\alpha\}_{\alpha \in A}$ let $WC(\{P_\alpha\}_{\alpha \in A})$ be the topological space with the underlying set $\bigvee_{\alpha \in A} \mathrm{Cone}(P_\alpha)$ (we denote the base point of it by p) so that a set U is open if and only if the following conditions are satisfied:

1. $U \cap Cone(P_\alpha)$ is open in $Cone(P_\alpha)$ for each α.
2. If $p \in U$, then there is $\alpha_0 \in A$ such that $P_\beta \subset U$ for all $\beta \geqslant \alpha_0$.

Proposition 9.2. *The space $X = WC(\{P_\alpha\}_{\alpha \in A})$ is paracompact. A shape system for $WC(\{P_\alpha\}_{\alpha \in A})$ is $(\bigvee_{\beta \geqslant \alpha} \Sigma(P_\beta), j_\alpha^{\alpha'}, A)$, where $j_\alpha^{\alpha'}$ is the natural inclusion.*

Proof. For each $\alpha \in A$ let K_α be the wedge of all cones $Cone(P_\beta)$, where β is not bigger than or equal α, and all suspensions $\Sigma(P_\beta)$ with $\beta \geqslant \alpha$. Let $\pi_\alpha : X \to K_\alpha$ be the projection so that P_β (the base of the $Cone(P_\beta)$) is mapped to the base point for $\beta \geqslant \alpha$. It is continuous by the following argument: Suppose V is an open subset of K_α and put $U = \pi_\alpha^{-1}(V)$. $U \cap Cone(P_\gamma)$ is open in $Cone(P_\gamma)$ for all γ as the projection $Cone(P_\gamma) \to \Sigma(P_\gamma)$ is continuous for all γ. If $p \in U$, then V contains the base point of K_α and U contains P_β for all $\beta \geqslant \alpha$.

To show that X is paracompact, it is sufficient to prove that, for any open cover $\{U_s\}_{s \in S}$ of X there is $\alpha \in A$ and an open cover $\{V_s\}_{s \in S}$ of K_α such that $\pi_\alpha^{-1}(V_s) \subset U_s$ for each $s \in S$. Pick $s(0) \in S$ with $p \in U_{s(0)}$ and choose $\alpha \in A$ with $P_\beta \subset U_{s(0)}$ for $\beta \geqslant \alpha$. Define $W_{s(0)} = U_{s(0)}$ and $W_s = U_s \setminus (\{p\} \cup \bigcup_{\beta \geqslant \alpha} P_\beta)$ for $s \neq s(0)$. Notice that for each $s \in S$ there is an open subset V_s of K_α such that $\pi_\alpha^{-1}(V_s) = W_s$. Since $\{W_s\}_{s \in S}$ is an open cover of X, $\{V_s\}_{s \in S}$ is an open cover of K_α and the proof of X being paracompact is completed.

To prove the second part of the proposition it suffices to show that $(K_\alpha, i_\alpha^{\alpha'}, A)$, where $i_\alpha^{\alpha'}$ is the natural projection, is the shape system of X. Since $i_\alpha^{\alpha'} \circ \pi_{\alpha'} = \pi_\alpha$ for $\alpha \leqslant \alpha'$, it suffices to show that the following two statements are valid:

(a) Given any map $f : X \to K$ from X to a CW complex K, there is $\alpha \in A$ and a map $g : K_\alpha \to K$ such that $g \circ \pi_\alpha$ is homotopic to f.

(b) Given $\alpha \in A$ and given two maps $f, g : K_\alpha \to K$ from K_α to a CW complex K such that $f \circ \pi_\alpha$ is homotopic to $g \circ \pi_\alpha$, there is $\beta \geqslant \alpha$ such that $f \circ i_\alpha^\beta$ is homotopic to $g \circ i_\alpha^\beta$.

Since every map to a CW complex is homotopic to a locally compact map (see [7]), we will reduce (a) and (b) to the case of f and g being locally compact.

Suppose $f:X \to K$ is a locally compact map from X to a CW complex K. Let C be a closed neighborhood of p in X such that $f(C)$ is contained in a compact subcomplex L of K containing the base point $*$ of K. Since L is locally contractible, there is a closed neighborhood D of p in X such that $f|D$ is homotopic to the constant map. By the Homotopy Extension Theorem for locally compact maps (see [7]), f is homotopic to $h:X \to K$ such that $h(D) = *$. As in the proof of paracompactness of X, there is $\alpha \in A$ and $g:K_\alpha \to K$ such that $h = g \circ \pi_\alpha$.

Suppose $\alpha \in A$ and suppose $f, g:K_\alpha \to K$ are two locally compact maps from K_α to a CW complex K such that $f \circ \pi_\alpha$ is homotopic to $g \circ \pi_\alpha$. As above, we may assume that both f and g are constant on some neighborhood of the basepoint of K_α. Also, we may assume that the homotopy H joining $f \circ \pi_\alpha$ and $g \circ \pi_\alpha$ is locally compact. By adjusting H we can make it constant on a neighborhood U of p. Find $\beta \geqslant \alpha$ such that $P_\gamma \subset U$ for all $\gamma \geqslant \beta$. As above, H can be factored through $K_\beta \times I$ which gives a homotopy joining $f \circ i_\alpha^\beta$ and $g \circ i_\alpha^\beta$. $\square$

Let us show that representatives of bimorphisms of the shape category have the property as in Proposition 8.2. $\square$

Proposition 9.3. *Let $f:X \to Y$ be a bimorphism in the shape category of pointed connected topological spaces. If f is represented by a level morphism $g:S(X) \to S(Y)$ of shape systems of X and Y, then g is an epimorphism of* pro-H_0 *and for every $\alpha \in I(S(X))$ there is $\beta > \alpha$ with the property that for any morphisms $a, b:\Sigma(P) \to S(X)_\beta$ of H_0, the equality $f_\beta \circ a = f_\beta \circ b$ implies $p(S(X))_\alpha^\beta \circ a = p(S(X))_\alpha^\beta \circ b$.*

Proof. Let D be the full subcategory of pro-H_0 whose objects are shape systems of pointed connected topological spaces. It is clear that g is a bimorphism of D. Notice that g is an epimorphism of pro-H_0. Indeed, if $u, v:Y \to Z$ satisfy $ug = vg$, then $(p(Z)_\alpha u)g = (p(Z)_\alpha v)g$ for all $\alpha \in I(Z)$. Now, Z_α is an object of D, so $p(Z)_\alpha u = p(Z)_\alpha v$ for all $\alpha \in I(Z)$ which is the same as $u = v$. It remains to show that for every $\alpha \in I(S(X))$ there is $\beta > \alpha$ with the property that for any morphisms $a, b:\Sigma(P) \to S(X)_\beta$ of H_0, the equality $f_\beta \circ a = f_\beta \circ b$ implies $p(S(X))_\alpha^\beta \circ a = p(S(X))_\alpha^\beta \circ b$. If that property does not hold, then, as in the proof of Proposition 2.4 in [9], there is a system $Z = (\bigvee_{\beta \geqslant \alpha} \Sigma(P_\beta), j_\alpha^{\alpha'}, A)$, where $j_\alpha^{\alpha'}$ is the natural inclusion, such that for some morphisms $u, v:Z \to S(X)$ one has $u \circ g = v \circ g$ but $u \neq v$. Since, by Proposition 9.2, Z is an object of D, one arrives at a contradiction. $\square$

In view of Proposition 8.2 one gets the following:

Theorem 9.4. *If $f:X \to Y$ is a bimorphism in the shape category of pointed connected topological spaces, then f is a weak equivalence.*

Proposition 9.5. *Let X be a pointed connected space. If $(K_\alpha, p_\alpha^\beta, A)$ is a shape system of X, then $(\Sigma K_\alpha, \Sigma p_\alpha^\beta, A)$ is a shape system of ΣX.*

Proof. Given $g:\Sigma X \to P \in ANR$ one has the adjoint map g' from X to the loop space ΩP. Also, given $g:X \to \Omega P$ one has the adjoint map from ΣX to P which will be denoted by g' as well.

Given $g:\Sigma X \to P \in ANR$ the adjoint $g':X \to \Omega P$ factors as $g' \sim h' \circ p_\alpha$ for some $h':K_\alpha \to \Omega P$. Now $g \sim h \circ \Sigma p_\alpha$, where $h:\Sigma K_\alpha \to P$ equals $(h')'$.

If $g,h:\Sigma K_\alpha \to P \in ANR$ so that $g \circ \Sigma p_\alpha \sim h \circ \Sigma p_\alpha$, then $g' \circ p_\alpha \sim h' \circ p_\alpha$ and there is $\beta > \alpha$ with $g' \circ p_\alpha^\beta \sim h' \circ p_\alpha^\beta$, i.e. $g \circ \Sigma p_\alpha^\beta \sim h \circ \Sigma p_\alpha^\beta$. $\square$

Theorem 9.6. *If $f:X \to Y$ is a bimorphism in the shape category of pointed topological spaces, then $f_*: \mathrm{Mor}(Z,X) \to \mathrm{Mor}(Z,Y)$ is a bijection for all sequentially movable spaces Z which are suspensions of some space Z'.*

Proof. Almost the same as in Theorem 8.5 if one uses Proposition 9.3. $\square$

Theorem 9.7. *Suppose $f:X \to Y$ is a bimorphism in the shape category of pointed topological spaces. If Y is sequentially movable, then f is an isomorphism in the following two cases:*

1. *Y is the suspension of a space Y'.*
2. *X is the suspension of a space X'.*

Proof. (1) Theorem 9.6 implies the existence of a left inverse of f, so f is an isomorphism by 1.2.

(2) f is a weak equivalence by Theorem 9.4, so Theorem 7.2 says that it is a strong epimorphism. Assume f is a level morphism of pro-H_0. Given $\alpha \in I(X)$ we can find $\beta > \alpha$ such that for any $a,b:\Sigma(P) \to X_\beta$ the condition $f_\beta \circ a = f_\beta \circ b$ implies $p(X)_\alpha^\beta \circ a = p(X)_\alpha^\beta \circ b$ (see Proposition 9.3). Since f is a strong epimorphism, there is $\gamma > \beta$ and $r:Y_\gamma \to X_\beta$ such that $f_\beta \circ r = p(Y)_\beta^\gamma$. Now, $f_\beta \circ (r \circ f_\gamma) = p(Y)_\beta^\gamma \circ f_\gamma = f_\beta \circ p(X)_\beta^\gamma$, so $p(X)_\alpha^\beta \circ r \circ f_\gamma = p(X)_\alpha^\beta \circ p(X)_\beta^\gamma = p(X)_\alpha^\gamma$ which proves that f is a strong monomorphism as well. By Corollary 3.14, f is an isomorphism. $\square$

Problem 9.8. If $f:X \to Y$ is a bimorphism in the shape category of pointed topological spaces, is f a bimorphism in pro-H_0?

Problem 9.9. If $f:X \to Y$ is a bimorphism of the shape category of pointed metric continua, is f a weak isomorphism? Is f an isomorphism?

Problem 9.10. If $f:X \to Y$ is a bimorphism of the shape category of pointed movable metric continua, is f a weak isomorphism?

10. Borsuk's problem and strong monomorphisms

The following question comes up naturally.

Problem 10.1. Let P be a finite connected pointed CW complex. Does P have stable strong images (respectively, stable strong subobjects) in pro-H_0?

In this section, we give partial answers to the part of Problem 10.1 dealing with strong subobjects and we point out that it is stronger than the following problem posed by Borsuk [2].

Problem 10.2 (K. Borsuk). Suppose X_n is a sequence of compact ANRs such that for each n there is a retraction $r_n : X_n \to X_{n+1}$. Is there a number m such that all retractions r_n are homotopy equivalences for $n > m$?

See [14,17,20,21] for partial solutions to the above problem.

If X is the inverse limit of the inverse sequence (X_n, i_n^m), where i_n^m is the inclusion for $m > n$, then the above problem is equivalent to stability of X. Indeed, in one direction it is quite obvious (if i_n^m are homotopy equivalences for n large enough) and in the other direction it follows that $\pi_k(i_n^m)$ are isomorphisms for large n and $k > \dim_{\mathrm{def}}(X_1)$, so i_n^m must be homotopy equivalences.

First let us show how one creates a strong monomorphism in the situation described by Borsuk's problem.

Proposition 10.3. *Suppose C is a category and X is an object of* pro-C. *If each $p(X)_\alpha^\beta$ has a left inverse then there is a strong monomorphism from X to $P \in \mathrm{Ob}(C)$.*

Proof. Pick $\gamma \in I(X)$. Given $\beta > \gamma$ let $r : X_\gamma \to X_\beta$ be the left inverse of $p(X)_\gamma^\beta$. Thus, $r \circ p(X)_\gamma^\beta = \mathrm{id}(X_\beta)$. This can be interpreted, in view of Proposition 3.7, as a proof that $p(Y)_\gamma : Y \to Y_\gamma$ is a strong monomorphism, where $I(Y) = \{\beta \in I(X) \mid \beta > \gamma\}$, $Y_\beta = X_\beta$, and $p(Y)_\tau^\sigma = p(X)_\tau^\sigma$ for all $\sigma, \tau \in I(Y)$. In other words, Y is a subsystem of X with $I(Y)$ cofinal in $I(X)$. Therefore, X and Y are isomorphic and X admits a strong monomorphism to X_γ. □

Problem 10.4. Suppose X is a pointed metric continuum. Is X uniformly movable if it admits a strong monomorphism to a compact polyhedron?

The last problem is stronger that Borsuk's one. The result below provides the justification of it. Indeed, every uniformly movable object X of pro-H_0 admits a strong epimorphism $Q \to X$ by Proposition 4.2.

Corollary 10.5. *Let X be an object of* pro-H_0. *If there exist polyhedra P, Q, a monomorphism $X \to P$, and a strong epimorphism $Q \to X$, then X is stable.*

Proof. By Proposition 4.7, X is dominated by an object of H_0 and [5] or [15, Theorem 4, p. 224] say that X is stable. □

The spaces of the type $WC(\{P_\alpha\}_{\alpha \in A})$ as in Definition 9.1, are not uniformly movable and they admit strong monomorphisms to non-compact polyhedra. On the other hand, if X is a pointed movable metric continuum and there is a monomorphism $X \to P \in \mathrm{Ob}(H_0)$ in the shape category of pointed movable metric continua, then X is stable.

It is well-known that Borsuk's problem has positive answer if X_1 is simply connected. Let us give a positive solution to Problem 10.4 if $\pi_1(P)$ is finite.

Theorem 10.6. *Suppose $f:X \to P$ is a strong monomorphism of* pro-H_0 *such that P is a compact CW complex. If* pro-$\pi_1(X)$ *is pro-finite or $\pi_1(P)$ is finite, then X is stable.*

Proof. Assume that f is a level morphism induced by $\{f_\alpha : X_\alpha \to P\}_{\alpha \in I(X)}$. Notice that the deformation dimension of X is bounded by $\dim(P)$ (see [5]). In view of results in [5] it suffices to prove that $\pi_n(X)$ is stable for all n.

Case 1: $\pi_1(P)$ is finite. In this case $\pi_n(f):\pi_n(X) \to \pi_n(P)$ is a strong monomorphism of pro-*Gr* and $\pi_n(P)$ is finitely generated and Abelian if $n \geqslant 2$. By Corollary 5.7, $\pi_n(X)$ is stable for all n.

Case 2: $\pi_1(X)$ is pro-finite. By results of [9] the pro-group $\pi_1(X)$ is stable (since it is pro-finite and admits a monomorphism to a group). Again, by results of [9], we may assume that all $\pi_1(X_\alpha)$ are finite and all $\pi_1(p(X)_\alpha^\beta)$ are isomorphisms. Since $\pi_1(f)$ is a strong monomorphism, we may assume that each $\pi_1(f_\alpha)$ has a left inverse. Therefore, we may think of $\pi_1(X)$ as a retract of $\pi_1(P)$ and the kernel of the retraction r can be killed by attaching 2-cells along $r(a) \cdot a^{-1}$ for every generator a of $\pi_1(P)$. This way one gets a finite CW complex Q containing P such that the composition $X \to P \to Q$ is a strong monomorphism and $\pi_1(Q)$ is finite. By Case 1, X is stable. □

Corollary 10.7. *Suppose X is an object of* pro-H_0. *If there is a strong monomorphism $f:X \to P$ and an epimorphism $g:Q \to X$ such that P is a finite CW complex and Q is a CW complex, then X is stable.*

A first step to solve Problem 10.4 would be the following:

Problem 10.8. Let X be an object of pro-H_0. Is X stable if there exist polyhedra P, Q, a strong monomorphism $X \to P$, and an epimorphism $Q \to X$?

The following problem is important because of possible applications to dynamical systems.

Problem 10.9. Suppose P be a finite polyhedron and $f:P \to P$ is a morphism of H_0. Let X be the tower in H_0 such that $X_n = P$ and $p(X)_n^{n+1} = f$ for each n. Is X stable if it is uniformly movable?

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Appendix A.

Let us prove categorical characterizations of strong monomorphisms and strong epimorphisms mentioned earlier. Notice that the category *Sets* of sets and functions is a category with direct sums, direct products, push-outs, and pull-backs. Therefore its dual category *Sets** has the same properties. Indeed, existence of push-outs (respectively, direct sums) in the dual category is equivalent to existence of pull-backs (respectively, direct products) in the original category.

First, we plan to show that both pro-*Sets* and pro-*Sets** have the property that every monomorphism (respectively, epimorphism) is a strong monomorphism (respectively, strong epimorphism).

Lemma A.1. *If*

$$\begin{tikzcd} X_\beta \arrow[r, "f_\beta"] \arrow[d, "p(X)^\beta_\alpha"'] & Y_\beta \arrow[d, "p(Y)^\beta_\alpha"] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \end{tikzcd}$$

is a commutative diagram of Sets, then the following conditions are equivalent:

(1) *There is* $r: Y_\beta \to X_\alpha$ *such that* $f_\alpha \circ r = p(Y)^\beta_\alpha$.
(2) *If* $u, v: Y_\alpha \to P$ *and* $u \circ f_\alpha = v \circ f_\alpha$, *then* $u \circ p(Y)^\beta_\alpha = v \circ p(Y)^\beta_\alpha$.

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$. Let $P = Y_\alpha/\mathrm{im}(f_\alpha)$, $u: Y_\alpha \to P$ is the projection, and $v: Y_\alpha \to P$ is the constant map to the point of P obtained by collapsing $\mathrm{im}(f_\alpha)$. Since $u \circ f_\alpha = v \circ f_\alpha$, $u \circ p(Y)^\beta_\alpha = v \circ p(Y)^\beta_\alpha$ which means exactly that $\mathrm{im}(p(Y)^\beta_\alpha) \subset \mathrm{im}(f_\alpha)$ in which case r exists. □

Corollary A.2. *Every epimorphism of* pro-*Sets is a strong epimorphism of* pro-*Sets*.

Proof. Assume $f = \{f_\alpha\}_{\alpha \in A}$ is a level morphism of pro-*Sets* and pick $\alpha \in A$. If f is an epimorphism, then Proposition 2.7 leads to a commutative diagram as in Lemma A.1. By Proposition 3.7, f is a strong epimorphism. □

Lemma A.3. *If*

$$\begin{tikzcd} X_\beta \arrow[r, "f_\beta"] \arrow[d, "p(X)^\beta_\alpha"'] & Y_\beta \arrow[d, "p(Y)^\beta_\alpha"] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \end{tikzcd}$$

is a commutative diagram of Sets, then the following conditions are equivalent:

1. *There is* $r : Y_\beta \to X_\alpha$ *such that* $r \circ f_\beta = p(X)^\beta_\alpha$.
2. *If* $u, v : P \to X_\beta$ *and* $f_\beta \circ u = f_\beta \circ v$, *then* $p(X)^\beta_\alpha \circ u = p(X)^\beta_\alpha \circ v$.

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$. Let P be the point-inverse of a point in Y_β under f_β. Let $u : P \to X_\beta$ be the inclusion and let $v : P \to X_\beta$ be any constant function to a point in P. Obviously, $f_\beta \circ u = f_\beta \circ v$, so $p(X)^\beta_\alpha \circ u = p(X)^\beta_\alpha \circ v$ which means that $p(X)^\beta_\alpha(P)$ contains at most one point. One defines $r : Y_\beta \to X_\alpha$ arbitrarily on $Y_\beta \setminus f_\beta(X_\beta)$ and $r(y) = p(X)^\beta_\alpha(f_\beta^{-1}(y))$ for $y \in \mathrm{im}(f_\beta)$. □

Corollary A.4. *Every monomorphism of* pro-*Sets is a strong monomorphism of* pro-*Sets*.

Proof. Assume $f = \{f_\alpha\}_{\alpha \in A}$ is a level morphism of pro-*Sets* and pick $\alpha \in A$. If f is a monomorphism, then Proposition 2.4 leads to a commutative diagram as in Lemma A.3. By Proposition 3.7, f is a strong monomorphism. □

Corollary A.5. *Every monomorphism* (*respectively*, *epimorphism*) *of* pro-*Sets** *is a strong monomorphism* (*respectively*, *strong epimorphism*) *of* pro-*Sets**.

Proof. Given a morphism $u : K \to L$ of *Sets**, we will denote by $\mu(u) : L \to K$ the corresponding function from L to K. Assume $f = \{f_\alpha\}_{\alpha \in A}$ is a level morphism of pro-*Sets** and pick $\alpha \in A$.

Case 1. f is a monomorphism of pro-*Sets**. By Proposition 2.4 there is $\beta > \alpha$ with the property that, for any $u, v : P \to X_\beta$, $f_\beta \circ u = f_\beta \circ u$ implies $p(X)^\beta_\alpha \circ u = p(X)^\beta_\alpha \circ v$. That is the same as saying that for any functions $u', v' : X_\beta \to P$ the equality $u' \circ \mu(f_\beta) = v' \circ \mu(f_\beta)$ implies $u' \circ \mu(p(X)^\beta_\alpha) = v' \circ \mu(p(X)^\beta_\alpha)$. By Lemma A.1 there is a function $r : X_\alpha \to Y_\beta$ such that $\mu(f_\beta) \circ r = \mu(p(X)^\beta_\alpha)$. If $g : Y_\beta \to X_\alpha$ is the morphism of *Sets** corresponding to r, then $r \circ f_\beta = p(X)^\beta_\alpha$ which proves that f is a strong monomorphism (see Proposition 3.7).

The case of epimorphisms can be proved similarly using Lemma A.3. □

Corollary A.6. *Suppose* $f : X \to Y$ *is a morphism of* pro-C. *If*, *for any covariant functor* $F : C \to D$, *the induced morphism* $F(f) : F(X) \to F(Y)$ *is a monomorphism* (*respectively*, *epimorphism*), *then* f *is a strong monomorphism* (*respectively*, *strong epimorphism*) *of* pro-C.

Proof. Assume $f = \{f_\alpha\}_{\alpha \in A}$ is a level morphism of pro-C and pick $\alpha \in A$.

Case 1: $F(f)$ is a monomorphism for any covariant functor $F : C \to D$. Consider $D = \mathit{Sets}^*$ and define $F(Z) = \mathrm{Mor}_C(Z, X_\alpha)$ regarded as a covariant functor from C to D. Since $F(f)$ is a strong monomorphism by Corollary A.5, Proposition 3.7 says there is $\beta > \alpha$ and a morphism $r : F(Y_\beta) \to F(X_\alpha)$ such that $r \circ F(f_\beta) = F(p(X)^\beta_\alpha)$. That implies $p(X)^\beta_\alpha$ belongs to the image of $F(f_\beta)$ (considered as a function from $\mathrm{Mor}_C(Y_\beta, X_\alpha)$

to $\mathrm{Mor}_C(X_\beta, X_\alpha)$) and there is $g: Y_\beta \to X_\alpha$ with $g \circ f_\beta = p(X)^\beta_\alpha$. Thus, see Proposition 3.7, f is a strong monomorphism.

Case 2: $F(f)$ is an epimorphism for any covariant functor $F: C \to D$. Consider $D = Sets$ and define $F(Z) = \mathrm{Mor}_{\text{pro-}C}(Y, Z)$ regarded as a covariant functor from C to D. Since $F(f)$ is a strong epimorphism by Corollary A.5, Proposition 3.7 says there is $\beta > \alpha$ and a function $r: F(Y_\beta) \to F(X_\alpha)$ such that $F(f_\alpha) \circ r = F(p(Y)^\beta_\alpha)$. Let $u = r(p(Y)_\beta)$. Now, $f_\alpha \circ u = p(Y)^\beta_\alpha \circ p(Y)_\beta = p(Y)_\alpha$. Thus, see Proposition 3.7, f is a strong epimorphism. □

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